Math 496 Final Report: Bredon Cohomology and Smith Theory

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1 Introduction

In algebraic topology, we always want to detect the topological properties of a given space through some topological invariant. In the study of the topological space with the topological group action, things will become more difficult than normal. For which, we developed a theory called equivariant homotopy theory. As one fundamental and famous theorem in the equivariant homotopy theory, Smith theory shows its power in the original problem we are interested in: the way to detect the topological properties of spaces with group action through topological invariant.

In 1940s, Paul Smith had proven several results related to the cohomology of the so-called G-CW complex and fixed points set, which is known as the **Smith Theory** [4]. We define that a space is an $\mathbb{F}_p$**-cohomology sphere** if there is an isomorphism of graded abelian group $H^*(X;\mathbb{F}_p) \cong H^*(S^n;\mathbb{F}_p)$ for some $n \geq 1$. The main result is given as follow:

Theorem 1 (**Smith**)**.** *Let G be a finite p-group, and X be a finite G-CW complex and X is a $\mathbb{F}_p$ cohmology n-sphere, then X^G is empty or is a $\mathbb{F}_p$ cohomology m-sphere for some $m \leq n$. If p is odd, then $n - m$ is even; and moreover if n is even, $X^G \neq \varnothing$.*

There are various way to prove the result, and some of them are rather tedious and highly technical, but here we will follow the proof originally given using modern language of Bredon cohomology, which is proved to be a very effective and elegant with much depth coming up.

Before we can use the power of Bredon cohomology, we need to do some works. The crucial part is showing existence of Bredon cohomology. So we need to construct the Bredon cohomology. There are two ways to construct it. The simplest and most energy-saving way is to construct a cochain complex in the category of coefficient system, and taking homology to the cochain complex. In order to do that, we need to first construct a chain complex through a coefficient system, namely $\underline{C}_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H;\mathbb{Z})$, where H_* is the ordinary homology, and $\underline{C}_n(X)$ is a functor from homotopy category of orbit G-spaces $ho(\mathrm{Orb_G})$ to Ab, which is a coefficient system. Then we take its dual $C^n_G(X;M) = \hom(\underline{C}_n(X), M)$ under some coefficient system M. Finally consider the coboundary map δ_n and compute $\ker d_n/\operatorname{im}\delta_{n+1}$ to get the Bredon cohomology $H^n_G(X;M) = \ker d_n/\operatorname{im}\delta_{n+1}$. The other way is concrete but rather abstract. We need to first construct a Eilenberg-MacLane space $K(G,n)$ through Elmendorf's theorem and bar construction, then connecting cohomology and homotopy groups through the Ω-spectrum $K(G.n)$ like what we did in non-equivariant homotopy.

In this paper, we will examine the construction of Bredon cohomology in either way in Section 3 and present a detailed proof towards Smith Theory in Section 4.

2 Preliminary

2.1 Categorical Setting

In this part, we will introduce the fundamental background needed in constructing the Bredon cohomology. First, we need some background knowledge in category theory.

Definition 1. A **category** $\mathcal{C}$ consists of

1. a class of objects $\mathrm{Obj}(\mathcal{C})$,
2. a class of morphisms $\mathrm{Mor}(\mathcal{C})$ between objects (a morphism is denoted by $f : a \to b$, where $a, b \in \mathrm{Obj}(\mathcal{C})$),
3. for every three objects a, b, c, a binary operator called composition $\hom(a,b) \times \hom(b,c) \to \hom(a,c)$, such that:
   (a) (Associativity) If $f : a \to b$, $g : b \to c$, $h : c \to d$, then $h \circ (g \circ f) = (h \circ g) \circ f$ and
   (b) (Identity) for every object x, there exists a morphism $1_x : x \to x$ (called identity morhism) such that for any $f : a \to x$, $g : x \to b$, we have $1_x \circ f = f$ and $g \circ 1_x = g$.

Example 1. The category of topological spaces Top. The category of G-spaces and G-maps GTop. The category of abelian groups Ab.

Definition 2. Let $\mathcal{C}, \mathcal{D}$ be categories. A **(covariant) functor** F is a mapping from $\mathcal{C}$ to $\mathcal{D}$ such that:

1. associate each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$,
2. associate each morphism $f : a \to b$ in $\mathcal{C}$ a morphism $F(f) : F(a) \to F(b)$ in $\mathcal{D}$ such that:
   (a) $F(1_X) = 1_{F(X)}$ for every object X in $\mathcal{C}$ and
   (b) for all morphisms $f : a \to b$, $g : b \to c$ in $\mathcal{C}$, $F(g \circ f) = F(g) \circ F(f)$.

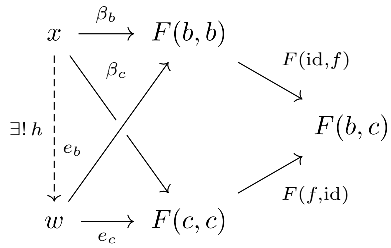
Definition 3. A **contravariant functor** F is a functor satisfies all conditions listed above with the last one replaced by

$$F(g \circ f) = F(f) \circ F(g).$$

Definition 4. Let $F : \mathcal{C}^{op} \times \mathcal{C} \to X$ be a functor. We define a **wedge** $e : w \to F$ to be an object $w \in \mathrm{Obj}(X)$ and a map $e_c : w \to F(c,c)$ for each $c \in \mathrm{Obj}(\mathcal{C})$, such that for any morphism $f : c \to c'$, the following diagram commutes:

$$\begin{array}{ccc} w & \xrightarrow{e_{c'}} & F(c',c') \\ \downarrow{\scriptstyle e_c} & & \downarrow{\scriptstyle F(f,\mathrm{id})} \\ F(c,c) & \xrightarrow{F(\mathrm{id},f)} & F(c,c') \end{array}$$

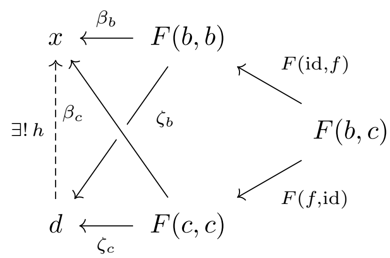
Definition 5. An **end** of F defined above is a pair $\langle w, e\rangle$, where $w \in \mathrm{Obj}(X)$, and $e : W \to F$ a wedge such that it is universal, i.e. for any pair $\langle x, \beta\rangle$, $\beta : x \to F$ a wedge, there exists a unique $h : x \to w$ with $\beta_a = e_a h$ for any $a \in \mathrm{Obj}(\mathcal{C})$ and any morphism $f : b \to c$, the following diagram commutes:



We denote $w = \int_{c\in\mathcal{C}} F(c,c)$.

We can define the concept of coend in a dual manner:

Definition 6. A **coend** is dual an end. Namely the pair $\langle d, \zeta\rangle$, denote $d = \int^{c\in\mathcal{C}} F(c,c)$, such that the following diagrams commutes in a dual manner as above: (we abuse the notation here)



Example 2. Let R be a commutative ring, and view it as a category. A right R-module A is an additive functor $A : R^{op} \to \mathrm{Ab}$, and a left R-module B is an additive functor $B : R \to \mathrm{Ab}$. Using the usual tensor product $\otimes_{\mathbb{Z}}$ in Ab, we have the bifunctor $A \otimes_{\mathbb{Z}} B : R^{op} \times R \to \mathrm{Ab}$, with coend

$$\int^{R} A \otimes_{\mathbb{Z}} B = A \otimes_R B,$$

a usual tensor product over R of a (left and right) R-module. From the example, we may observe that the coend is really a coequalizer in the target category.

Example 3. Extend the Example 2 above. Let $\mathcal{A}$ be a monoidal category, $F : \mathcal{C}^{op} \to \mathcal{A}$ and $G : \mathcal{C} \to \mathcal{A}$. External tensor product defines a bifunctor $F \otimes G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{A}$, and the coend gives the usual functor tensor product:

$$\int^{\mathcal{C}} F \otimes G = F \otimes_{\mathcal{C}} G.$$

2.2 Geometric Setting

From now on, we will always denote Δ to be the category of sets $[n] = \{0, 1, 2, \cdots, n\}$, with the morphisms are order-preserving maps between these sets.

Definition 7. A **simplicial object** in a category $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$. When $\mathcal{C} = \mathrm{Set}$ or Top, we call this functor a **simplicial set** or a **simplicial space**, respectively. In this case, we have two following important definitions:

1. **Face map** of a simplicial set $X : \Delta^{op} \to \mathrm{Set}$ are the images in that simplicial set of the morphisms $\delta^{n,0}, \cdots, \delta^{n,n} : [n-1] \to [n]$, where $\delta^{n,i}$ is the only order-preserving injection $[n-1] \to [n]$ that misses i. Denote these face maps by $d_{n,0}, \cdots, d_{n,n}$, with $d_{n,i} : X_n \to X_{n-1}$ and $X(\delta^{n,i}) = d_{n,i}$. When there is no confusion, we simplify them by $d_0, \cdots, d_n$.

2. **Degeneracy maps** of a simplicial set $X : \Delta^{op} \to \text{Set}$ are the images in that simplicial set of the morphisms $\sigma^{n,0}, \cdots, \sigma^{n,n} : [n+1] \to [n]$, where $\sigma^{n,i}$ is the only order-preserving surjection $[n+1] \to [n]$ that hits i. twice. Denote these face maps by $s_{n,0}, \cdots, s_{n,n}$, with $s_{n,i} : X_n \to X_{n+1}$ and $X(\sigma^{n,i}) = s_{n,i}$. When there is no confusion, we simplify them by $s_0, \cdots, s_n$.

Remark 1. The defined maps satisfy the following simplicial identities:

1. $d_i d_j = d_{j-1} d_i$, for $0 \leq i < j \leq n$.
2. $d_i s_j = s_{j-1} d_i$, for $0 \leq i < j \leq n$.
3. $d_i s_j = \text{id}$, for $i = j$ or $i = j+1$.
4. $d_i s_j = s_j d_{i-1}$, for $0 \leq j+1 < i \leq n$.
5. $s_i s_j = s_{j+1} s_i$, for $0 \leq i \leq j \leq n$.

Moreover, if a sequence of simplicial sets X_n together with face and degeneracy maps d_i, s_i that satisfy the simplicial identities above, then they define a unique simplicial set X satisfying all conditions.

Example 4. With face and degeneracy maps defined above, we obtain a covariant functor $\Delta_* : \Delta \to \text{Top}$ in the usual way. Explicitly, $\Delta_*([n]) = \{(x_1, \cdots, x_n) : x_i \leq x_{i+1}\} \subset \mathbb{R}^n$.

Example 5. For a simplicial space $X_* : \Delta^{op} \to \text{Top}$, we have a functor

$$X_* \times \Delta_* : \Delta^{op} \times \Delta \to \text{Top}.$$

Definition 8. The **geometric realization** of X_* is the coend $|X_*| = \int^{\Delta} X_* \times \Delta_*$.

Example 6. If X is a simplicial G-space (i.e. $X_* : \Delta^{op} \to \text{GTop}$), then $|X_*|$ inherits a G-action such that $|X_*|^H = |X_*^H|$ for all $H \subset G$.

Remark 2. The reason the "geometric realization" has its name is because, by construction, $|X_*|$ is a topological space given by the quotient

$$|X_*| = \coprod_n X_n \times \Delta_n / \sim,$$

where the equivalence relation "$\sim$" is given by, for all $[k] \xrightarrow{f} [\ell]$ in Δ, the points $(x, f_* p) \in X_n \times \Delta_\ell$ are identified with $(f^* x, p) \in X_n \times \Delta_k$.

Remark 3. The geometric realization is actually a functor $|-| : \Delta^{op} \to \text{Top}$, and $|X_*| = X_* \otimes_\Delta \Delta_*$ by Example 3.

Now let $\mathcal{D}$ be a small category. Define $B_*(\mathcal{D})$ to be the set of n-tuples $\underline{f} = (f_1, \cdots, f_n)$ of composable arrows of $\mathcal{D}$, depicted by

$$x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} x_2 \xleftarrow{f_3} \cdots \xleftarrow{f_n} x_n,$$

where $x_i \in \text{Obj}(\mathcal{D})$, and topologized as a subspace of the n-fold product of the total morphism space $\coprod \mathcal{D}(x, x')$.

Definition 9. We say $B_*(\mathcal{D})$ is the **nerve** of $\mathcal{D}$, if we make it a simplicial set structure.

Remark 4. We'll define what is mean to have a simplicial structure. By Remark 1, it suffices to find the face and degeneracy maps.

Explicitly, the face map $d_i : B_k(\mathcal{D}) \to B_{k-1}(\mathcal{D})$, where $B_k(\mathcal{D}) = \{\underline{f^{(k)}} = (f_1^{(k)}, \cdots, f_k^{(k)}) : x_0 \xleftarrow{f_1^{(k)}} x_1 \xleftarrow{f_2^{(k)}} \cdots \xleftarrow{f_k^{(k)}} x_k\}$ and $B_{k-1}(\mathcal{D}) = \{\underline{f^{(k-1)}} = (f_1^{(k-1)}, \cdots, f_k^{(k-1)}) : x_0 \xleftarrow{f_1^{(k-1)}} x_1 \xleftarrow{f_2^{(k-1)}} \cdots \xleftarrow{f_{i-1}^{(k-1)}} x_{i-1} \xleftarrow{\delta} x_{i+1} \xleftarrow{f_{i+1}^{(k-1)}} \cdots \xleftarrow{f_k^{(k-1)}} x_k\}$, and $\delta = f_i^{(k)} \circ f_{i+1}^{(k)}$. That is to say, d_i composes the morphisms $f_i^{(k)}$ and $f_{i+1}^{(k)}$ into a new morphism δ, yielding a $(k-1)$-tuple for every k-tuple.

Similarly, we have the degeneracy map $s_i : B_k(\mathcal{D}) \to B_{k+1}(\mathcal{D})$, given by inserting identity morphism at object x_i, yielding a $(k+1)$-tuple for every k-tuple.

Note 1. The geometric realization of $B_*(\mathcal{D})$, $|B_*(\mathcal{D})| =: BD$, is called the **classifying space**.

Let $T : \mathcal{D} \to \mathrm{Top}$ be a continuous contravariant functor, and $S : \mathcal{D} \to \mathrm{Top}$ be a continuous covariant functor.

Definition 10. The **bar construction** is defined to be

$$B(T, \mathcal{D}, S) = |B_*(T, \mathcal{D}, S)|,$$

where $B_*(T, \mathcal{D}, S)$ is the simplicial space whose set of n-simplices is $\{(t, \underline{f}, s) : t \in T(x_0), \underline{f} \in B_n(\mathcal{D}), s \in S(x_n)\}$, topologized as a subspace of the product $(\coprod(T(x))) \times (\coprod \mathcal{D}(x, x'))^n \times (\coprod(S(x)))$.

Remark 5. $B_0 = \coprod T(x) \times S(x)$. In the same way we did in Remark 4, we can give a simplicial structure to $B_*(T, \mathcal{D}, S)$ to make the preceding Definition well-defined, with zeroth and last face using the evaluation of the functors T and S. Also as we did in Example 3, $B(T, \mathcal{D}, S) = T \otimes_{\mathcal{D}} S$.

Example 7. There is a GTop version. Let $T, S : \mathcal{D} \to \mathrm{GTop}$, then $B_*(T, \mathcal{D}, S)$ is a simplicial G-space and $B(T, \mathcal{D}, S)$ is a G-space, with $B(T, \mathcal{D}, S)^H = B(T, \mathcal{D}, S^H)$.

Lemma 1. Let $x \in \mathrm{Obj}(\mathcal{D})$ be an object. Then it induces a homotopy equivalence

$$\eta : B(T, \mathcal{D}, \hom(x, -)) \to T(x).$$

In a general setting, we have the following definition in categorical sense:

Definition 11. Let $S : \mathcal{D} \to \mathrm{Top}$ be a covariant functor. We define its **homotopy colimit** to be

$$\operatorname{hocolim} S \coloneqq B(*, \mathcal{D}, S),$$

where $* : \mathcal{D} \to \mathrm{Top}$ is a trivial functor mapping every object to a one-point space.

It can be easily imagined there is a definition of homotopy limit in duality. We will briefly introduce the concept and skip all details here since they won't affect our main goal.

We have a **cosimplicial space** in dual to the simplicial space $B_*(T, \mathcal{D}, S)$. Use the notation before, the cosimplicial space $B^*(T, \mathcal{D}, S)$ is the set of all n-cosimplices $\coprod_{\underline{f} \in B_n(\mathcal{D})}(T(x_0) \times S(x_n))$, $n \in \mathbb{N}$, and $B^*(T, \mathcal{D}, S)$ is topologized as a subspace of $\mathrm{Map}(B_n(\mathcal{D}), \coprod T(x) \times S(x)) = (T(x_0) \times S(x_n))^{B_n(\mathcal{D})}$ with compact-open topology. We can also define the coface and codegeneracy maps in duality to make it "actually" a cosimplicial space. Finally, we define its geometric realization (called **totalization**) to be

$$\mathrm{Tot}\, B^*(T, \mathcal{D}, S) = |B^*(T, \mathcal{D}, S)| = \int_{\mathcal{D}} \mathrm{Map}(T, S).$$

Definition 12. Let $T : \mathcal{D} \to \mathrm{Top}$ be a contravariant functor. We define its **homotopy limit** to be

$$\operatorname{holim} T \coloneqq \mathrm{Tot}\, B^*(T, \mathcal{D}, S),$$

where $* : \mathcal{D} \to \mathrm{Top}$ is a trivial functor mapping every object to a one-point space.

2.3 Basic Concepts in Equivariant Homotopy Theory

Let G be a topological group. In this part, we will introduce some basic concepts in equivariant homotopy theory.

Definition 13. A **G-space** is a topological space with the group action $G \curvearrowright X$ such that $ex = x$ and $g(g'x) = (gg')x$ for any $x \in X$.

Definition 14. A map $f : G \times X \to X$ is a **G-map** if $gf(x) = f(gx)$ for any $g \in G$.

The category of G-maps and G-spaces forms a subcategory of Top, denoted by GTop. We assume for simplicity that subgroups of G are closed. For $H \subset G$, we define the **H-fixed points set** $X^H = \{x : hx = x\, \forall h \in H\}$. Define the Weyl groups NH to be the normalizer of H in G, $WH = NH/H$, it is clear that X^H is a WH-space. In equivariant theory, the orbit G/H plays the role of points, and the set of G-maps from G/H to itself can be identified with the group WH.

Like what did in Top, we can define the **G-CW complex** X to be the union of G-spaces (the n-skeleton) X^n such that X^0 is the disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching G-cells $G/H \times D^{n+1}$ along the attaching G-maps $G/H \times S^n \to X^n$. Namely, such that the following diagram commutes:

$$\begin{array}{ccc} \coprod_\alpha G/H_\alpha \times S^n & \xrightarrow{attaching} & X^n \\ \downarrow & & \downarrow \\ \coprod_\alpha G/H_\alpha \times D^{n+1} & \xrightarrow{attaching} & X^{n+1} \end{array}$$

Same stories, such as Whitehead Theorem, Cellular Approximation, CW Approximation, happen in analogue of those in non-equivariant homotopy theory, and we will not discuss the details here (would mention them as long as we need). Readers may refer to [1] or a brilliant online note [5].

Define the **Orbit category** associated to a group G, denoted by $\mathrm{Orb_G}$, is a category with objects are G-orbits G/H (where $H \subset G$) and morphisms are G-maps. The homotopy category of $\mathrm{Orb_G}$ is denoted by $ho(\mathrm{Orb_G})$, which is obtained by Bousfield localization of $\mathrm{Orb_G}$.

Definition 15. A **coefficient system** $\mathcal{A}$ is a contravariant functor $\mathcal{A} : ho(\mathrm{Orb_G}) \to \mathrm{Ab}$, where Ab is the category of abelian groups.

Remark 6. One can regard it as a continuous contravariant functor from $\mathrm{Orb_G}$ to Ab.

Definition 16. Consider a finite field $\mathbb{F}_p$, the **argumentation ideal** I of $\mathbb{F}_p[G]$ (a group ring) is defined to be the kernel of the map $\mathbb{F}_p[G] \to \mathbb{F}_p$ sending all $g \in G$ to 1.

Remark 7. *I^n is n^{th} power of I, and can be viewed as a coefficient system whose value on G is I^n and on points is 0. Also we have $I^p = 0$ and $I^n/I^{n+1} \cong \mathbb{F}_p$ for $0 \le n \le p-1$.*

Definition 17. The **reduced Euler characteristic** for a space X is $\tilde{\chi}(X) = \sum_i (-1)^i \mathrm{rank}\, \tilde{H}^n(X)$.

3 Construction of Bredon Cohomology

3.1 Basic Construction

Now we are ready to construct the Bredon Cohomology. But first we need to introduce a useful category, which would be useful later on.

Definition 18. Define the **Orbit category** associated to a group G, denoted by $\mathrm{Orb_G}$, is a category with objects are G-orbits G/H (where $H \subset G$) and morphisms are G-maps.

Recall that a contravariant functor from category $\mathcal{C}$ to Set is called a **presheaf**. An evident example is when $\mathcal{C} = \Delta$, then the simplicial set is a presheaf. Let $\mathfrak{P}(\mathrm{Orb_G})$ be the category of presheaves $P : \mathrm{Orb_G}^{op} \to$ Top.

Let $X \in \mathrm{Obj(GTop)}$. Consider the "fixed point functor" $X^{(-)}$ for every $H \subset G$, that is to say, it sends every H to a fixed point space X^H. This gives a functor $\phi : \mathrm{GTop} \to \mathfrak{P}(\mathrm{Orb_G})$, sending X to $X^{(-)}$.

Lemma 2. *ϕ has a left adjoint $\theta : \mathfrak{P}(\mathrm{Orb_G}) \to \mathrm{GTop}$, i.e.*

$$\mathfrak{P}(\mathrm{Orb_G})(P, \phi X) \cong \mathrm{GTop}(\theta P, X),$$

for every presheaf $P \in \mathrm{Obj}(\mathfrak{P}(\mathrm{Orb_G}))$ and G-space $X \in \mathrm{Obj(GTop)}$.

Proof. Define $\theta : \mathfrak{P}(\mathrm{Orb_G}) \to \mathrm{GTop}$, sending a presheaf P to $P(G/e)$, on which G-action is induced by the group action on the orbit G/e. That means, for every G-space X, $\phi(X) = X^{(-)}$, and $\theta(\phi(X)) = X^e = X$. So $\theta\phi = \mathrm{id}$.

On the other hand, the quotient map $G \to G/H$ induces $\varphi : P(G/H) \to P(G/e)^H$, where $P(G/e)^H$ is the space of morphisms $P(G/H) \to P(G/e)$, and these maps together specify a natural map $\rho : P \to \phi\theta P$. Passage from $\phi : P \to \phi X$ to $\theta\phi : \theta P \to X$ is a bijection whose inverse sends $f : \theta P \to X$ to $\phi f \circ \rho$. ☐

Theorem 2 (Elmendorf). There is a functor $\psi : \mathfrak{P}(\mathrm{Orb_G}) \to \mathrm{GTop}$ and a natural transformation $\varepsilon : \phi\psi \to \mathrm{id}$, such that $\varepsilon : (\psi S)^H \to S(G/H)$ is a homotopy equivalence. Moreover, if X has the type of a G-CW complex, then there is a natural bijection

$$[X, \psi S]_{\mathrm{GTop}} \cong [\phi X, S]_{\mathfrak{P}(\mathrm{Orb_G})}.$$

Proof. Let $T : \mathrm{Orb_G} \to \mathrm{GTop}$ be the covariant functor given by G/H (as a orbit) $\mapsto G/H$ (as a G-space), and $\mathrm{Orb_G}(G/H, G/K) \mapsto \mathrm{GTop}(G/H, G/K)$. Let $S \in \mathrm{Obj}(\mathfrak{P}(\mathrm{Orb_G}))$, we define $\psi S = B(S, \mathrm{Orb_G}, T)$. We have

$$T^H(G/K) = (G/K)^H = \mathrm{GTop}(G/H, G/K) = \mathrm{Orb_G}(G/H, G/K).$$

Note $B(S, \mathrm{Orb_G}, T) = S \otimes_{\mathrm{Orb_G}} T$, and $B(S, \mathrm{Orb_G}, T)^H = B(S, \mathrm{Orb_G}, T^H)$. So we have a homotopy equivalence by Lemma 1:

$$\varepsilon : (\psi S)^H \to S(G/H),$$

which defines a natural transformation $\varepsilon : \phi\psi \to \mathrm{id}$. From $\theta\phi = \mathrm{id}$, we see $\theta\varepsilon : \theta\phi\psi S = \psi S \mapsto \theta S$ is a weak homotopy equivalence of G-space for any S. Write $S = \phi X$, this gives a weak equivalence $\theta\varepsilon : \psi\phi X \mapsto X$. Hence $\psi\phi X$ has the G-CW complex of X has. In this case, Whitehead Theorem implies $\theta\varepsilon$ is actually a homotopy equivalence. Thus it has an inverse $(\theta\varepsilon)^{-1}$.

Let $\alpha : [X, \psi S]_{\mathrm{GTop}} \to [\phi X, S]_{\mathfrak{P}(\mathrm{Orb_G})}$ by $\alpha(f) = \varepsilon \circ \phi f$; and $\beta : [\phi X, S]_{\mathfrak{P}(\mathrm{Orb_G})} \to [X, \psi S]_{\mathrm{GTop}}$ by $\beta(g) = \psi g \circ (\theta\varepsilon)^{-1}$. Then one can check

$$\alpha\beta(g) = \alpha(\psi g \circ (\theta\varepsilon)^{-1}) = \varepsilon \circ \phi(\psi g \circ (\theta\varepsilon)^{-1}) \simeq g$$
$$\beta\alpha(f) = \beta(\varepsilon \circ \phi f) = \psi(\varepsilon \circ \phi f) \circ (\theta\varepsilon)^{-1} \simeq \psi\varepsilon \circ (\theta\varepsilon)^{-1} \circ f$$

From the fact $\psi\varepsilon$ is weak homotopy equivalence, we have $\beta\alpha$ bijection by Whitehead Theorem, which implies α, β are inverse bijections. ☐

Now we're ready to construct the Bredon cohomology. Let B be the classifying space functor and M be a coefficient system (a contravariant functor from $ho(\mathrm{Orb_G})$ to Ab). Then $B^n \circ M \in \mathfrak{P}(\mathrm{Orb_G})$. Then from Elmendorf's Theorem, $\varepsilon : \psi(B^n \circ M)^H \to (B^n \circ M)(G/H)$ is a homotopy equivalence.

Definition 19. The **Eilenberg-MacLane space** is defined by

$$K(M, n) = \psi(B^n \circ M),$$

where $\psi : \mathfrak{P}(\mathrm{Orb_G}) \to \mathrm{GTop}$ defined above is the geometric realization $\psi(-) = B(-, \mathrm{Orb_G}, T)$, and $T : \mathrm{Orb_G} \to \mathrm{GTop}$ is the covariant functor sending a orbit G/H to the corresponding G-space G/H.

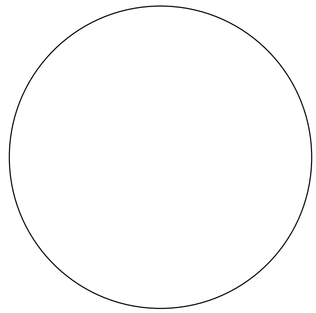
Finally, in analogue to what we did in connecting the cohomology theory with homotopy groups via Ω-spectrum of Eilenberg-MacLane spaces in the non-equivariant homotopy case, we have the following definition:

Definition 20. The **Bredon cohomology** $\tilde{H}^n_G$ is defined to be

$$\tilde{H}^n_G(X; M) \coloneqq [X, K(M, n)]_{\mathrm{GTop}} \cong [\phi X, B^n \circ M]_{\mathfrak{P}(\mathrm{Orb_G})}.$$

3.2 Elementary Example

We now examine an elementary example. Consider the simplest case: the circle S^1, with C_2 acting on it. C_2 is the group $\{1, -1\}$, where -1 is the antipodal map.



Then we can regard this circle (a C_2-space) is decomposed into:

$$\begin{gathered} \text{0-cell: } C_2/\{1\} \times D^0, \\ \text{1-cell: } C_2/\{1\} \times D^1. \end{gathered}$$

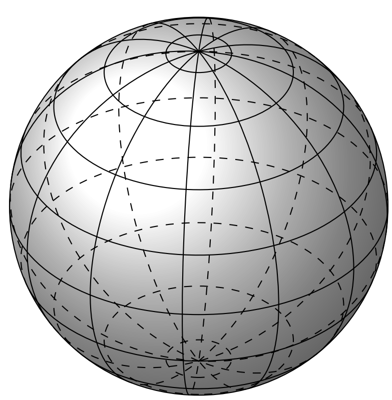
The attaching maps can be identified through the following diagram: (x_0, x_1 are two antipodal points on the circle, and 0-skeleton is given by $X^0 = \{x_0, x_1\}$)

$$\begin{array}{ccc} C_2/\{1\} \times S^0 & \xrightarrow{attaching} & X^0 \\ \big\downarrow & & \big\downarrow \\ C_2/\{1\} \times D^1 & \xrightarrow{attaching} & X^1 = S^1 \end{array}$$

And we identify points through $x_0 \sim \{1\} \times \{0\}$, $x_1 \sim \{1\} \times \{1\}$, $x_0 \sim -1\{1\} \times \{1\}$ and $x_1 \sim -1\{1\} \times \{0\}$. This gives the C_2-CW complex structure of $X = S^1$.

In general, we can move our view to n-dimensional case. Consider C_2-CW complex $X = S^n$. In the same way we did above, we can identify that the C_2-CW complex structure of X is given by

$$\begin{gathered} n\text{-cell: } C_2/\{1\} \times D^n, \\ \text{Lower dimensional cells: } C_2/C_2 \times \text{ cells of } S^{n-1} \text{ in non-equivariant case }. \end{gathered}$$



The attaching maps can be identified through the following diagram:

$$\begin{array}{ccc} C_2/\{1\} \times S^{n-1} & \xrightarrow{attaching} & X^{n-1} = S^{n-1} \\ \downarrow & & \downarrow \\ C_2/\{1\} \times D^n & \xrightarrow{attaching} & X^n = S^n \end{array}$$

with attaching map given by $C_2/\{1\} \times S^{n-1} \to S^{n-1}$, $(g\{1\}, x) \mapsto x$. Use this information, we are able to give a concrete description of its Bredon cohomology. Let M be a coefficient system. For topological group C_2 and one of its subgroup H, we have $\underline{C}_k(S^n)(C_2/H) = H_k((S^n)^H, (S^{n-1})^H; \mathbb{Z})$. Notice from the definition of C_2 (H is either trivial or C_2 itself) and equivariant cellular structure of S^n, we have that (note $(S^n)^{C_2} = *$) $\underline{C}_k(S^n)(C_2/H) = \mathbb{Z}^2$ for $C_2/H = C_2$ and $k \leq n$, and 0 elsewhere. This implies that the cochain complex is given by

$$C^k_{C_2}(S^n) = \hom(\underline{C}_k(S^n), M) = \hom(\mathbb{Z}^2, M). \tag{$*$}$$

If $M(C_2) = 0$, then $\hom(\mathbb{Z}^2, M) \equiv 0$, the Bredon cohomology always vanishes. So we assume $M(C_2) \neq 0$. For simplicity, we assume $M(C_2) = \mathbb{Z}$. It is equivalent to regard M here is just the '$\mathbb{Z}$-coefficient' in the non-equivariant situation. Therefore from $(*)$, we get

$$C^k_{C_2}(S^n) = \begin{cases} \mathbb{Z} & , k \leq n \\ 0 & , \text{otherwise} \end{cases}$$

Note from previous discussion, the attaching map for k-skeleton has two parts: one is through $1 \in C_2$, which has degree 1. While the other is given by $-1 \in C_2$, which has degree $(-1)^k$. Hence the coboundary map $\delta_k : C^k_{C_2}(S^n) \to C^{k+1}_{C_2}(S^n)$ is given by $\delta_k(\cdot) = (1 + (-1)^k)(\cdot)$. Therefore it is not hard to compute Bredon cohomology via its definition (kernel/image):

$$H^k_{C_2}(S^n; M) = \begin{cases} \mathbb{Z} & , k = n, k = 0 \text{ if } k \text{ odd} \\ C_2 & , k \leq n \text{ if } k \text{ even} \\ 0 & , \text{otherwise} \end{cases}$$

This coincides the usual cohomology of $\mathbb{R}P^n$. This is valid, because if we consider C_2 acting on S^n, this is really the definition of real projective space in n-dimension. This reveals a connection between equivariant homotopy and usual homotopy.

4 Smith Theory

In this section, unless stated otherwise, we will always assume G is a finite topological p-group and the G-CW complex X is finite. We restate the main result as follow:

Theorem 3 (Smith). *X is $\mathbb{F}_p$-acyclic implies X^G is empty or $\mathbb{F}_p$-acyclic.*

Explicitly, if X is a $\mathbb{F}_p$-cohomology n-sphere, then X^G is a $\mathbb{F}_p$-cohomology m-sphere for some $m \leq n$. If p is odd, then $n - m$ is even; and moreover if n is even, $X^G \neq \varnothing$.

We will use Bredon Cohomoly to tackle this theorem. Nevertheless, the critical tools we need is the axioms of cohomology theory for Bredon cohomology $H^*_G(X; \mathcal{A})$ with some coefficient system $\mathcal{A}$:

Theorem 4. *For G a topological group, $\mathcal{A}$ a coefficient system. $\exists!$ functors $\tilde{H}^*_G(-; \mathcal{A}) : \mathrm{GTop}^{op}_* \to \mathrm{Ab}$ together with an isomorphism $\tilde{H}^n_G(X; \mathcal{A}) \xrightarrow{\cong} \tilde{H}^{n+1}_G(\Sigma X; \mathcal{A})$ satisfying the following axioms:*

1. (Addivity) $X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induces isomorphisms

$$\tilde{H}^*_G(\bigvee_\alpha X_\alpha; \mathcal{A}) \cong \prod_\alpha \tilde{H}^*_G(X_\alpha; \mathcal{A})$$

2. (Exactness) If $X \xrightarrow{f} Y \to Cf$ is a cofiber sequence, then the sequence $\tilde{H}^n_G(Cf;\mathcal{A}) \to \tilde{H}^n_G(Y;\mathcal{A}) \to \tilde{H}^n_G(X;\mathcal{A})$ is exact.
3. (Weak Equivalences) $\tilde{H}^*_G(-;\mathcal{A})$ sends weak equivalences to isomorphisms.
4. (Dimension) Let $G/H \in \mathrm{Obj}(\mathrm{Orb_G})$, then $\tilde{H}^n_G(G/H;\mathcal{A}) = \mathcal{A}(G/H)$ for $n = 0$, and 0 otherwise.

We can check he Bredon cohomology H^*_G constructed in Section 3 truly satisfies the axioms. Now we can present the proof.

Proof of Theorem 3. Observe that $X^G = (X^H)^{G/H}$. By Sylow's theorem, one can induct on the order of group G. So the whole proof reduces to the situation $G = \mathbb{Z}/p$. WLOG, we assume $G = \mathbb{Z}/p$.

Now a exact sequence of coefficient systems $0 \to L \to M \to N \to 0$ induces a long exact sequence

$$\cdots \to H^n_G(X;L) \to H^n_G(X;M) \to H^n_G(X;N) \to H^{n+1}_G(X;L) \to \cdots$$

by Bockstein homomorphism. Define $FX = X/X^G$ with the free action of G on it.

Goal 1. We need to construct L, M, N such that we can connect Bredon cohomology to ordinary cohomology:

$$H^n_G(X;L) \cong \tilde{H}^n(FX/G;\mathbb{F}_p) \tag{1}$$
$$H^n_G(X;M) \cong \tilde{H}^n(X;\mathbb{F}_p) \tag{2}$$
$$H^n_G(X;N) \cong \tilde{H}^n(X^G;\mathbb{F}_p). \tag{3}$$

This part is rather easy. Just applying the dimension axiom and taking $X = G$ and $X = *$, we have

$$\begin{array}{ll} L(G) = \mathbb{F}_p, & L(*) = 0, \\ N(G) = 0, & N(*) = \mathbb{F}_p, \\ M(G) = \mathbb{F}_p[G], & M(*) = \mathbb{F}_p. \end{array}$$

Notice that $G^G = \varnothing$, $*^G = *$, $F(*)/G = \varnothing$ and $FG/G = G_+/G$, where G_+ is G disjoint with a fixed basepoint. Now let I be the argumentation ideal of $\mathbb{F}_p[G]$. Clearly we have $M/I \cong \mathbb{F}_p$ and $I^{p-1} \cong \mathbb{F}_p \cong L$, by looking at the evaluations at $X = G$ and $X = *$.

From previous discussion, we have the following exact sequences of coefficient systems ($1 \le n < p-1$):

$$0 \to I \to M \to N \oplus L \to 0 \tag{4}$$
$$0 \to L \to M \to N \oplus I \to 0 \tag{5}$$
$$0 \to I^{n+1} \to I^n \to L \to 0 \tag{6}$$

The exactness is easy to show by looking at the evaluations. Sequences (4) and (5) coincide when $p = 2$. Plus, Sequences (4) and (5) induce long exact sequences accordingly (use (1)$\sim$(3), and all ordinary cohomology are in $\mathbb{F}_p$-coefficient):

$$\cdots \to H^q_G(X;I) \to H^q(X) \to \tilde{H}^q(FX/G) \oplus H^q(X^G) \to H^{q+1}_G(X;I) \to \cdots \tag{4'}$$
$$\cdots \to \tilde{H}^q(FX/G) \to H^q(X) \to H^q_G(X;I) \oplus H^q(X^G) \to \tilde{H}^{q+1}(FX/G) \to \cdots \tag{5'}$$

Define $a_q = \operatorname{rank} H^q(X^G;\mathbb{F}_p)$, $b_q = \operatorname{rank} H^q(X;\mathbb{F}_p)$, $c_q = \operatorname{rank} \tilde{H}^q(FX/G;\mathbb{F}_p)$, $d_q = \operatorname{rank} H^q_G(X;I)$. It's evident that $c_q = d_q$ when $p = 2$.

Exactness of (4′) and (5′) implies that

$$a_q + c_q \le b_q + d_{q+1} \tag{4''}$$
$$a_q + d_q \le b_q + c_{q+1} \tag{5''}$$

Adding (4″) for q even and (5″) for q odd together, we have for $r, q \ge 0$,

$$c_q + \sum_{i=q}^{q+r} a_i \le c_{q+r+1} + \sum_{i=q}^{q+r} b_i, \tag{7}$$

and by taking $q=0, r\to\infty$ (note X is finite, so c_q vanishes eventually), we have

$$\sum a_i \leq \sum b_i.$$

Now we have a simple fact stated as follow:

Lemma 3. For the long exact sequence of finite dimensional vector spaces

$$\to C^{n-1} \to A^n \to B^n \xrightarrow{\phi_n} C^n \to A^{n+1} \cdots$$

we have $\chi(B) = \chi(A) + \chi(C)$.

This is easy to check by directly writing out the definition of $\chi(A), \chi(B), \chi(C)$ and then comparing, and noticing that $\dim B^n = \dim(\operatorname{im}\phi_n) + \dim(\ker\phi_n)$ (similar for A, C).

Use either $(4')$ or $(5')$, we can write

$$\chi(X) = \chi_I(X) + \chi(X^G) + \tilde{\chi}(FX/G),$$

where $\chi_I(X)$ is the Euler characteristic associated to Bredon cohomology of X with coefficient I. The sequence (6), by similar argument, induces

$$\chi_{I^n}(X) = \chi_{I^{n+1}}(X) + \tilde{\chi}(FX/G).$$

Using the fact $I^{p-1}\cong L$, and Summing them together over $1 \leq n < p-1$, we obtain

$$\chi(X) = \chi(X^G) + p\tilde{\chi}(FX/G),$$

which implies

$$\chi(X) \equiv \chi(X^G) \mod p. \tag{8}$$

Goal 2. Now we need to use the previous results to prove the theorem.

Since X is a $\mathbb{F}_p$-cohomology n-sphere, $\sum b_i = 2$, and hence $\sum a_i = 0$ or 1 or 2. The possibility of 1 is ruled out by equation (8). So $\sum a_i = 0$ or 2. If it is 0, then X^G is empty. Now assume $\sum a_i = 2$, then X^G is another $\mathbb{F}_p$-cohomology m-sphere. We have $m \leq n$ by taking $r\to\infty$ and $p = n+1$ in equation (7).

If p is odd, then $p > 2$, and (8) holds iff $\chi(X) = \chi(X^G)$, which implies $n-m$ is even. Moreover, if n is even, $\chi(X) = 2$, and thus $\chi(X^G) = 2$, implying $X^G \neq \varnothing$. □

The converse of Smith theorem also holds partly, proved by Lowell Jones (1971) and Robert Oliver (1975). We will state them without proofs:

Theorem 5 (Jones)**.** If X is $\mathbb{Z}/n$-acyclic, then $X = Y^{\mathbb{Z}/n}$ for a contractible Y with semi-free $\mathbb{Z}/n$-action.

Theorem 6 (Oliver)**.** Let G be a group with order not a prime power, then there is a number n_G such that $X = Y^G$ for a contractible Y with G-action iff $\chi(X) \equiv 1 \mod n_G$.

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