A Taste of Bott Periodicity Theorem

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Abstract

This paper gives a brief introduction to the fundamental theorem of topological K-theory, the Bott Periodicity Theorem. We will examine the motivation of the theorem and give a stretch of the proof of it. Some basic computational examples will be provided. Background of vector bundles is assumed.

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- Kirsche, Z. Topological K-Theory.
- LECTURE 7:K-THEORY GROUPS OF THE SPHERES

1 Introduction

In this paper, we always denote

$$\mathrm{Vect}(X) = \{\text{isomorphism classes of complex vector bundles over X}\},$$

where X is assumed to be compact Hausdorff.

To begin with, we first define the concept of K-theory group:

Definition 1.1. *Let A be an abelian semi-group. The K**-theory group** $K(A)$ is defined through the following universal property:*

For $\alpha : A \to K(A)$ a semi-group homomorphism, $\forall G$ a group, and $\gamma : A \to G$ a semi-group homomophism, $\exists! \beta : K(A) \to G$ a group homomorphism such that $\gamma = \beta\alpha$.

With this definition, we would expect an abelian group structure arising from the abelian semi-group. To see this actually works, Let $F(A)$ be a free abelian group generated by the elements of the given abelian semi-group A, then modulo by the elements of the form $a + a' - a \oplus a'$, where $\oplus$ is the addition in A and $a, a' \in A$. Denote this new quotient group by $K(A)$, easy to check it satisfies the definition above. Moreover, if A is a semi-ring (i.e. it possesses a multiplication which is distributive over the addition A), then $K(A)$ is a ring, called the K**-theorey ring**.

Now, note that for any space X, $\mathrm{Vect}(X)$ has the commutative semi-ring structure (where addition is the Whitney sum and the multiplication is the tensor product). So we may import the following notation:

Notation 1.2. *$K(X) = K(\mathrm{Vect}(X))$.*

One can check this is a commutative ring since it arises from a commutative semi-ring.

Remark 1.3. *Let $\underline{n}$ be trivial bundle over X of rank n. Then elements in $K(X)$ is of the form*

$$\begin{aligned}[E] - [F] &= [E] + [G] - ([F] + [G]) \\ &= [E \oplus G] - [F \oplus G] \\ &= [H] - [\underline{n}],\end{aligned}$$

where E, F are vector bundles over X, G is a vector bundle over X s.t. $F \oplus G = \underline{n}$, $H = E \oplus G$.

Example 1.4. *Compute $K(*)$, where $*$ is a point. Then we know that the only vector bundles over $*$ is trivial bundles of rank n, where $n \in \mathbb{N}$. Therefore, one can check $\mathrm{Vect}(*) \cong \mathbb{Z}$, hence $K(*) = \mathbb{Z}$.*

Example 1.5. *Compute $K(S^0)$. Note that S^0 is just two points. Assigning separate bundles to each component gives that $K(S^0) = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$.*

With these in hand, we can show that K has the functoriality: it maps from the category of topological spaces **Top** to the category of commutative rings **CRing**.

We can define the **external product**: $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ by $\mu(a \otimes b) = p_X^*(a)p_Y^*(b)$, where p_X, p_Y are projections of $X \times Y$ onto X and Y, respectively. This is a ring homomorphism, since for $a, b \in K(X)$, $c, d \in K(Y)$, $(a \otimes b)(c \otimes d) = ac \otimes bd$ (tensor product of ring is again a ring), and we have $\mu((a \otimes b)(c \otimes d)) = \mu(ac \otimes bd) = p_X^*(ac)p_Y^*(bd) = p_X^*(a)p_X^*(c)p_Y^*(b)p_Y^*(d) = p_X^*(a)p_Y^*(b)p_X^*(c)p_Y^*(d) = \mu(a \otimes b)\mu(c \otimes d)$.

Let $Y = S^2 = \mathbb{C}\mathrm{P}^1$. Now what can we tell about this μ then? The best result should be μ is an isomorphism. This is really the case of Bott periodicity theorem. In the following sections, we will devote to give a stretch proof of this isomorphism.

Theorem 1.6 (Bott Periodicity Theorem)**.** *For any compact Hausdorff space X, the map*

$$\mu : K(X) \otimes K(S^2) \to K(X \times S^2) \tag{1}$$

is a isomorphism of rings.

Let H be the canonical line bundle over $S^2 = \mathbb{C}\mathrm{P}^1$, then we know from [Hatcher, Example 1.13] that $(H \otimes H) \oplus \underline{1} \cong H \oplus H$, where we can write it in $K(S^2)$ in the form $H^2 + 1 = 2H$, i.e. $(H - 1)^2 = 0$. So we have a natural ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$.

Corollary 1.7. *The map*

$$\mu_0 : \mathbb{Z}[H]/(H-1)^2 \to K(S^2) \tag{2}$$

is an isomorphism of rings.

This theorem is fundamental in the topological K-theory. From the theorem and the corollary, one can compute a bunch of K-theory rings.

2 Setting the Stage

First note the fact that vector bundles over S^2 corresponds to homotopy classes of maps $S^1 \to \mathrm{GL}(n, \mathbb{C})$ (similar idea to Čech cocycle), which we'll call the clutching functions. To prove the theorem, one need to generalize this construction to create the vector bundle over $X \times S^2$ by means of the "generalized clutching function", which we'll give a sketch below.

For a vector fundle $\pi : E \to X$, let $f \in \mathrm{Aut}(E \times S^1)$ of the product bundle $\pi \times \mathbb{1} : E \times S^1 \to X \times S^1$. This gives an isomorphism $f(x, z) : \pi^{-1}(x) \to \pi^{-1}(x)$, for all $(x, z) \in X \times S^1$. From this E and f one can construct a new bundle over $X \times S^2$ by taking two copies of $E \times D^2$ and identifying the subspace $E \times S^1$ via f. Denote this new bundle as $[E, f]$, and call f the **clutching function**.

Proposition 2.1. *If $f_t : E \times S^1 \to E \times S^1$ is a homotopy, $t \in [0, 1]$, then $[E, f_0] \cong [E, f_1]$.*

Example 2.2. *Taking $X = *$. Then $[\underline{1}, z] \cong H$. Moreover, we have $[\underline{1}, z^n] \cong H^n$. If we define $H^{-1} = [\underline{1}, z^{-1}]$, then previous statement holds for $n \in \mathbb{Z}$, which is justified by the fact that $H \otimes H^{-1} = \underline{1}$.*

Example 2.3. *For arbitrary bundle E, we have $[E, z^n] \cong \mu(E \otimes H^n)$ for $n \in Z$. In general, we have $[E, z^n f] \cong [E, f] \otimes \hat{H}^n$, where $\hat{H}^n$ denotes the pullback of H^n via the projection $X \times S^2 \to S^2$.*

Example 2.4. *Every vector bundle $E' \to X \times S^2$ is isomorphic to some $[E, f]$. To see this, let the unit circle $S^1 \subset S^2 = \hat{\mathbb{C}}$ decompose S^2 into two disks D_0 and D_∞. Let E_α for $\alpha = 0, \infty$ be the restriction of E' over $X \times D_\alpha$, with E the restriction of E' over $X \times \{1\}$. Then the projection $X \times D_\alpha \to X \times \{1\}$ is homotopic to the identity map of $X \times D_\alpha$, which implies E_α is isomorphic to the pullback of E by the projection, and this pullback is $E \times D_\alpha$. This gives an isomorphism $h_\alpha : E_\alpha \to E \times D_\alpha$. The clutching function is then $f = h_0 h_\infty^{-1}$.*

We may assume a clutching function f is normalized to be the identity over $X \times \{1\}$ since we may normalize any isomorphism $h_\alpha : E_\alpha \to E \times D_\alpha$ by composing it over each $X \times \{z\}$ with the inverse of its restriction over $X \times \{1\}$. Any two choices of these normalized h_α for a given bundle are homotopic since they differ by a map g_α from D_α to the automorphisms of E, with $g_\alpha(1) = \mathbb{1}$. Such g_α is again homotopic to the constant map $\mathbb{1}$ by composing it with a deformation retraction from D_α to $\{1\}$. This gives that any two choices of f_0 and f_1 of normalized clutching functions are joined by a homotopy of normalized clutching functions f_t.

So how we examine the clutching function, which is a continuous function, for an arbitrary bundle?

Example 2.5. *Consider the bundle $\xi : X \times S^1 \to S^1$, then ξ can be seen as a $\mathbb{C}$-valued function on $X \times S^1$ which we denote by z (here S^1 is identified with complex numbers of unit modulus). Hence we can consider functions on $X \times S^1$, which are finite Laurant series in z with coefficients are functions on X. Namely, $\sum_{k=-n}^{n} a_k(x) z^k$, where $a_k : X \to \mathbb{C}$.*

Remark 2.6. *Function in the above example can be used to approximate functions on S^1.*

3 Examine clutching functions

3.1 Approximation

In general, the strategy of proving the theorem is to approximate an arbitrary clutching function by some simpler one. One efficient way is the generalization of Example 2.5: consider the so-called **Laurant (polynomial) clutching function**, which has the form $l(x,z)=\sum_{|i|\leq n} a_i(x)z^i$, where $a_i : E \to E$ restricts to a linear transformation $a_i(x)$ in each fiber $p^{-1}(x)$. We have the following result:

Proposition 3.1. *$\forall\,[E,f]\cong[E,l]$ for some Laurant clutching function l. l_0 and l_1 are homotopic via Laurant clutching function homotopy $l_t=\sum_i a_i(x,t)z^i$.*

In order to prove it, we need to seek a way to approximate the continuous function f defined on the compact set X by the Laurant clutching function $l(x,z)=\sum_{|n|\leq N} a_n(x)z^n$. Take $z=\mathrm{e}^{i\theta}$. Then motivated by Fourier series, we define that:

Definition 3.2. ***Fourier coefficient*** *of f is defined by*

$$a_n(x)=\frac{1}{2\pi i}\int_0^{2\pi} f(x,\mathrm{e}^{i\theta})\mathrm{e}^{-in\theta}d\theta.$$

Let $u(x,r,\theta)=\sum_{n\in\mathbb{Z}} a_n(x)r^{|n|}\mathrm{e}^{in\theta}$, where $0<r<1$. This r controls the series to be convergent. From the theory of Fourier series, we should have:

Lemma 3.3. *$u(x,r,\theta)\to f(x,\mathrm{e}^{i\theta})$ uniformly in x and θ, as $r\to 1$.*

With this lemma in hand, we are ready to prove Proposition 3.1.

Proof. Endow a Hermitian inner product structure on E, we can define the norm on $\mathrm{End}(E\times S^1)$ that $\|\alpha\|=\sup_{|v|=1}|\alpha(v)|$ (can verify this actually gives a metric), which making $\mathrm{End}(E\times S^1)$ a vector space. Therefore, we can view $\mathrm{End}(E\times S^1)$ as a topological space w.r.t. the metric, and so subspace $\mathrm{Aut}(E\times S^1)$ is open in this space since it is the preimage of $(0,\infty)$ under continuous map $\mathrm{End}(E\times S^1)\to[0,\infty)$ by $\alpha\to\inf_{(x,z)\in X\times S^1}|\det(\alpha(x,z))|$. Hence it suffices to prove that Laurent clutching functions are dense in $\mathrm{End}(E\times S^1)$, because a sufficiently close Laurent clutching functions approximation l to f will then be homotopic to f via the linear homotopy $tl+(1-t)f$. The other half of the statement follows similarly by approximating a homotopy from l_0 to l_1, viewed as an element of $\mathrm{Aut}(E\times S^1\times I)$ by a Laurant clutching homotopy l'_t, then combining this with linear homotopies from l_0 to l'_0 and l_1 to l'_1 (Proposition 2.1).

Now to show Laurent clutching functions are dense in $\mathrm{End}(E\times S^1)$. First choose open sets U_i covering X together with trivialization $h_i:\pi^{-1}(U_i)\to U_i\times\mathbb{C}^{n_i}$. We may assume h_i takes the chosen inner product in $\pi^{-1}(U_i)$ to the standard inner product in $\mathbb{C}^{n_i}$, by applying the Gram-Schmidt process to h_i^{-1} of the standard basis vectors. Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$ and $\operatorname{supp}\rho_i\subset U_i$. Then $\forall$ linear map $f(x,z)\in\mathrm{End}(E\times S^1)$, it can be viewed as matrices via h_i, with entries of these matrices defining functions $\operatorname{supp}\rho_i\times S^1\to\mathbb{C}$. By Lemma 3.3 we can find Laurent clutching functions $l_i(x,z)$ (viewed as matrices) with entries uniformly approximating those of $f(x,z)$ for

$x \in \operatorname{supp} \rho_i$. Thus it follows that l_i approximates f in the $\| \cdot \|$ norm defined above. Let $l = \sum_i \rho_i l_i$, we conclude our proof.

$\square$

A Laurent clutching function can be written $l = z^{-m} q$ for a polynomial clutching function q, and then we have $[E, l] \cong [E, q] \otimes \hat{H}^{-m}$ by Example 2.3. The following lemma gives reduction of clutching functions to linear clutching functions.

Lemma 3.4. *Let q be a polynomial clutching function of degree at most n, then $[E, q] \oplus [nE, \mathbb{1}] \cong [(n+1)E, L^n q]$ for a linear clutching function $L^n q$.*

3.2 Linear clutching functions

For the linear clutching functions, we can write them in the form $a(x)z + b(x)$. Then we have the following result:

Lemma 3.5. *Given a bundle $[E, a(x)z + b(x)]$, there is a splitting $E \cong E_+ \oplus E_-$ with $[E, a(x)z + b(x)] \cong [E_+, \mathbb{1}] \oplus [E_-, z]$.*

Remark 3.6. *This splitting preserves the direct sum in the sense that for a sum $[E_1 \oplus E_2, (a_1 z + b_1) \oplus (a_2 z + b_2)]$, we have $(E_1 \oplus E_2)_\pm \cong (E_1)_\pm \oplus (E_2)_\pm$.*

The proof of this lemma is rather tedious and hard (see [Hatcher, Proposition 2.7]), but it will do no harm to skip it. Thus, We will always assume this lemma from now on.

4 Proof of Periodicity Theorem

With so many tools in hand, we are ready to prove the theorem. We address the question of showing that the following homomorphism

$$\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$$

is an isomorphism.

4.1 Surjectivity of μ

Propositon 3.1 tells us that in $K(X \times S^2)$ we have

$$\begin{aligned}
[E, f] &= [E, z^{-m} q] = [E, q] \otimes \hat{H}^{-m} \\
&= [(n+1)E, L^n q] \otimes \hat{H}^{-m} - [nE, \mathbb{1}] \otimes \hat{H}^{-m} && \text{(By Lemma 3.4)} \\
&= [((n+1)E)_+, \mathbb{1}] \otimes \hat{H}^{-m} + [((n+1)E)_-, z] \otimes \hat{H}^{-m} - [nE, \mathbb{1}] \otimes \hat{H}^{-m} && \text{(By Lemma 3.5)} \\
&= \mu(((n+1)E)_+ \otimes H^{-m}) + \mu(((n+1)E)_- \otimes H^{1-m}) - \mu(nE \otimes H^{-m}).
\end{aligned}$$

Therefore, μ is surjective.

4.2 Injectivity of μ

In order to prove the injectivity of μ, we want to construct a map $\nu : K(X \times S^2) \to K(X) \otimes \mathbb{Z}[H]/(H-1)^2$ s.t. $\nu\mu = \mathbb{1}$.

IDEA 4.1. *Define $\nu([E,f])$ to be the linear combination of $E \times H^k$ and $((n+1)E)_\pm \otimes H^k$.*

In order to construct ν, we first need the following results:

Lemma 4.2. *Let $\deg q \leq n$, then*

1. $[(n+2)E, L^{n+1}q] \cong [(n+1)E, L^n q] \oplus [E, \mathbb{1}]$,
2. $[(n+2)E, L^{n+1}(zq)] \cong [(n+1)E, L^n q] \oplus [E, z]$

Proof. The matrix representation of $L^{n+1}q$ is given by

$$\begin{bmatrix} 1 & -z & 0 & \cdots & 0 & 0 \\ 0 & 1 & -z & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -z \\ 0 & a_n & a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix}$$

Adding first column z times to second column, yielding the matrix representation of $[(n+1)E, L^n q] \oplus [E, \mathbb{1}]$, which proves "1". Similarly, one can prove "2" in the same pattern. □

After we obtain the formula in Lemma 4.2, we have the observations:

Lemma 4.3. *For splitting of $[E, \mathbb{1}]$ and $[E, z]$, we have*

1. *For $[E, \mathbb{1}]$ the summand E_- is 0 and E_+ is E.*
2. *For $[E, z]$ the summand E_+ is 0 and E_- is E.*

These are not hard to prove by using some facts in the proof of Lemma 3.5, which we'll skip the details.

Under Lemma 4.2 "1" and Lemma 4.3 "1", we have $((n+2)E)_- \cong ((n+1)E)_-$, by the fact in Remark 3.6 that the splitting preserves the "$-$". Thus "$-$" part summand is independent of n.

Notation 4.4. *Let*

$$\nu([E, z^{-m}q]) = ((n+1)E)_- \otimes (H-1) + E \otimes H^{-m} \in K(X) \otimes \mathbb{Z}[H]/(H-1)^2,$$

where $\deg q \leq n$.

Claim 4.5. *This is well-defined.*

Proof. We've checked this definition is independent of the choice of n. Now to check it doesn't depend on m. Considering $z^{-m-1}(zq)$, we have by Lemma 4.2 "2" and Lemma 4.3 "2" that

$$\nu([E, z^{-m-1}(zq)]) = ((n+1)E)_- \otimes (H-1) + E \otimes (H-1) + E \otimes H^{-m-1}$$

Since $(H-1)^2 = 0$, and $H(H-1) = H - 1 = H^2 - H$, inductively we have $H - 1 = H^{-m-1}(H-1) = H^{-m} - H^{-m-1}$. Thus

$$\begin{aligned}\nu([E, z^{-m-1}(zq)]) &= ((n+1)E)_- \otimes (H-1) + E \otimes (H^{-m} - H^{-m-1}) + E \otimes H^{-m-1} \\ &= ((n+1)E)_- \otimes (H-1) + E \otimes H^{-m} \\ &= \nu([E, z^{-m}q]).\end{aligned}$$

Finally, note that by Proposition 3.1 we know that every bundle over $X \times S^2$ is uniquely of the form $[E, l]$ for some Laurent polynomial clutching function l, up to homotopy. We conclude this ν is dependent only on E but not the clutching function $z^{-m}q$. Hence we finish the proof of well-definiteness of ν. □

It is easy to check that ν defined above takes sums to sums, so we can extend it to a homomorphism $\nu : K(X \times S^2) \to K(X) \otimes \mathbb{Z}[H]/(H-1)^2$.

The last step is to check $\nu\mu = \mathbb{1}$. Note that in $\mathbb{Z}[H]/(H-1)^2$ we have $(H-1)^2 = 0$, which implies $H + H^{-1} = 2$. So one can regard that it is generated by $1, H^{-1}$. From

$$\begin{aligned}\nu\mu(E \otimes H^{-m}) = \nu([E, z^{-m}]) &= E_- \otimes (H-1) + E \otimes H^{-m} \\ &= E \otimes H^{-m},\end{aligned}$$

since we have $q = \mathbb{1}$, so $n = 0$ and by Lemma 4.3 "1" $E_- = 0$. Hence we've checked that $\nu\mu = \mathbb{1}$.

So far, we've done the proof that $\mu : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2$ is an isomorphism. And this is really the proof Bott Periodicity Theorem we're seeking for.

5 A simple observation of modern Bott Periodicity Theorem

In the modern vision of Bott Periodicity Theorem, we don't want our statement of theorem involving the $K(S^2)$, and it totally depends on the given original space X. In order to do that, we first introduce the concept of reduced K-theory group.

Definition 5.1. *Let $x_0 \in X$. The **reduced K-theory group** of X, denoted as $\tilde{K}(X)$, is defined to be $\tilde{K}(X) = \ker\{K(X) \to (x_0)\}$, which induced by the obvious inclusion map $\{x_0\} \hookrightarrow X$.*

Remark 5.2. *In practice, we can regard $K(X) = \tilde{K}(X) \times \mathbb{Z}$, by Example 1.4.*

Example 5.3. *From Example 1.4, we know that $\tilde{K}(*) = 0$.*

Example 5.4. *From Example 1.5, we know that $\tilde{K}(S^0) = \mathbb{Z}$.*

Example 5.5. *By definition, we have a split short exact sequence*

$$0 \to \tilde{K}(S^2) \to K(S^2) \xrightarrow{g} K(*) \to 0.$$

We've proved that $K(*) = \mathbb{Z}$, and $K(S^2) = \mathbb{Z}[H]/(H-1)^2$ is generated by $1, H$. The homomorphism g sends $aH + b$ to $a + b$. Hence by exactness, $\ker g = \tilde{K}(S^2) = \mathbb{Z}$.

To begin with, we introduce a fundamental property of reduced K-theory group without proof:

Proposition 5.6. *For X compact Hausdorff, $A \subset X$ a closed subspace, then the inclusion and the quotient map $A \overset{i}{\hookrightarrow} X \overset{\pi}{\to} X/A$ induce an exact sequence $\tilde{K}(X/A) \overset{\pi^*}{\to} \tilde{K}(X) \overset{i^*}{\to} \tilde{K}(A)$.*

This gives rise to

Corollary 5.7. *We have a long exact sequence associated to the short exact sequence defined above:*

$$\cdots \to \tilde{K}(SX) \to \tilde{K}(SA) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A),$$

where SX means the suspension of X, namely $SX = (X \times I)/\sim$, where $(x, i) \sim (y, i)$ for all $x, y \in X$ and $i = 0, 1$.

From Proposition 5.6 and Corollary 5.7, one can choose a pair $(X \times Y, X \vee Y)$ to obtain

$$\tilde{K}(S(X \times Y)) \to \tilde{K}(SX) \oplus \tilde{K}(SY) \to \tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y),$$

which implies the splitting

$$\tilde{K}(X \times Y) \cong \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y).$$

From which, we obtain the modern vision of our main theorem:

Theorem 5.8 (Bott Periodicity Theorem)**.** *Let X be compact Hausdorff, then we have an isomorphism*

$$\tilde{K}(X) \to \tilde{K}(\Sigma^2 X),$$

where ΣX is the reduced suspension of X.

Proof. By definition of reduced K-theory group and the discussion above, we have the commutative diagram:

$$\begin{array}{ccccc} K(X) \otimes K(S^2) & \cong & \tilde{K}(X) \otimes \tilde{K}(S^2) & \oplus & \tilde{K}(X) \oplus \tilde{K}(S^2) \oplus \mathbb{Z} \\ \downarrow & & \downarrow & & \downarrow \\ K(X \times S^2) & \cong & \tilde{K}(\Sigma^2 X) & \oplus & \tilde{K}(X) \oplus \tilde{K}(S^2) \oplus \mathbb{Z}, \end{array}$$

The left homomorphism and the right homomorphism are isomophisms, so the middle one is also an isomorphism. From Example 5.5 that $\tilde{K}(S^2) = \mathbb{Z}$ we're done the proof. $\square$

Finally, we have a simple application:

Corollary 5.9. *For S^n, we have $\tilde{K}(S^{2n}) = \mathbb{Z}$ and $\tilde{K}(S^{2n+1}) = 0$.*