Goal 1. Intro to slice spectral sequence.

2. Compute π_{**} of $H\mathbb{Z}$, kgl, kq over $\mathbb{C}$.

3. If time permits, compute " " " $\mathbb{R}$.

---

§1. Slice spectral sequences.

IDEA: Atiyah-Hirzebruch SS, but in the sense of motivic setting.

Classically, AHSS goes like:

1° Let E be a spectrum. Consider its Postnikov tower

$$\begin{array}{ccc} & & \vdots \\ \Sigma^n H\pi_n E & \longrightarrow & \tau_{\leq n} E \\ & & \downarrow \\ \Sigma^{n-1} H\pi_{n-1} E & \longrightarrow & \tau_{\leq n-1} E \\ & & \downarrow \\ & & \vdots \end{array}$$

where $\tau_{\leq n} : Sp \longrightarrow Sp_{\leq n}$

$Sp_{\leq n} := \{ X \in Sp : \pi_k X = 0 \text{ for } k > n \}$.

2° Applying $- \wedge X_+$ and taking $\pi_*(-)$, gets an exact couple and thus a SS

$$E_2^{s,t} = H^s(X, \pi_{-t} E)$$

$\Rightarrow \quad E^{s+t}(X)$

(assume $E \simeq \lim_n \tau_{\leq n} E$. so strongly convergent)

Issue : 1° need to define "$\tau_{\leq n}$" in motivic setting.

2° need to define "motivic CW spectra"

3° need to take care of the convergence.

<u>Def</u> Let $\mathcal{C}$ be an ∞-cat.

$\mathcal{F}$ = collection of objs of $\mathcal{C}$ as a set.

Define $\langle \mathcal{F} \rangle$ = smallest full subcat s.t.

1) $\mathcal{F} \subseteq \langle \mathcal{F} \rangle$

2) if $X \simeq F \in \langle \mathcal{F} \rangle$. then $X \in \langle \mathcal{F} \rangle$

3) $\langle \mathcal{F} \rangle$ is closed under htpy colims.

Call these are $\mathcal{F}$ - cellular obj.

e.g. $\mathcal{C}$ = ∞ - cat of spaces.

$\mathcal{F} = \{ S^n : n \in \mathbb{Z} \}$.

$\langle \mathcal{F} \rangle = \mathcal{C}$.

e.g. cellular motivic spectra :

$\mathcal{C} = SH(k)$

$\mathcal{F} = \{ S^{p,q} : p, q \in \mathbb{Z} \}$

$\langle \mathcal{F} \rangle =: SH(k)_c$.

Now to answer question $1°$.

Let $\mathcal{C}$ be presentable ∞-cat. Consider a chain of objs in $\mathcal{C}$:

$$\cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \cdots \subseteq \mathcal{C}.$$

$$\rightsquigarrow \quad \cdots \subseteq \langle \mathcal{F}_{n-1} \rangle \subseteq \langle \mathcal{F}_n \rangle \subseteq \langle \mathcal{F}_{n+1} \rangle \subseteq \cdots \subseteq \mathcal{C}.$$

At each stage, one has a right adjoint (Lurie)

$$\begin{tikzcd} \langle \mathcal{F}_{n-1} \rangle & & \langle \mathcal{F}_n \rangle \arrow[ll] \\ & \mathcal{C} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Let $\mathcal{C} = SH(k)_c$, $\mathcal{F}_q = \{ S^{p,q'} : p \in \mathbb{Z}, q' \geq q \}$

each $\langle \mathcal{F}_q \rangle$ is triangulated. Now define

$$f_q : SH(k)_c \longrightarrow \langle \mathcal{F}_q \rangle$$

and $\quad f_{q+1} \longrightarrow f_q \longrightarrow s_q \quad$ cofib seq.

then for each $E \in SH(k)_c$, get

$$\begin{tikzcd} \vdots \arrow[d, phantom] & \\ f_{q+1}E \arrow[r] \arrow[d] & s_{q+1}E \\ f_q E \arrow[r] & s_q E \end{tikzcd}$$

$\vdots$

this is the slice tower, and applying $- \wedge X_+$ & π_{**}. get the slice spectral sequence.

$$E^1_{m,q,n} = \pi_{m,n}(s_q E \wedge X_+)$$
$$\Rightarrow \pi_{m,n}(E \wedge X_+)$$

not rigorous, but enough for our case.

$$d^r : E^r_{m,q,n} \longrightarrow E^r_{m-1,q+r,n}.$$

Rk 1) effective slice filtration:

$\mathcal{C} = SH(k)$

$\mathcal{F}_q = \{ \Sigma^{p,q} \Sigma^\infty_+ X : p \in \mathbb{Z},\ X \in Sm_k \}$

$\rightsquigarrow$ effective slice spectral sequence (ESSS)

2) very effective slice filtration:

$\mathcal{C} = SH(k)_c$

$\mathcal{F}_q = \{ S^{2q',q'} : q' \geq q \}$.

$\rightsquigarrow$ very effective slice spectral sequence (VSSS)

3) $s_0 \mathbb{1} \simeq H\mathbb{Z} \simeq s_0 KGL$

$f_1 H\mathbb{Z} \simeq * \rightsquigarrow f_0 H\mathbb{Z} \simeq s_0 H\mathbb{Z}$

4) $s_q KGL \simeq \Sigma^{2q,q} s_0 KGL \simeq \Sigma^{2q,q} H\mathbb{Z}$.

$f_q KGL \simeq \Sigma^{2q,q} f_0 KGL$.

Write $kgl := f_0 KGL$. Bott: $\Sigma^{2,1} KGL \simeq KGL$

5) Bott $\Sigma^{8,4} KQ \simeq KQ$.

Aside (Hornbostel, 05)

$$KQ_i := \begin{cases} aKQ_f, & i \equiv 0 \mod 4 \\ aUSp_f, & i \equiv 1 \mod 4 \\ aKSp_f, & i \equiv 2 \mod 4 \\ aU_f, & i \equiv 3 \mod 4 \end{cases}$$

where 1° aKQ_f fibrant replacement of aKQ = Nis. sheafification of UKQ, where

$$UKQ : Sm_k^{op} \longrightarrow Sm_k^{op}$$

$$UKQ_n(X) = \mathrm{Hom}_{\mathcal{H}(k)}(S^n \wedge X_+, aKQ_f)$$

2° $USp(X) := \mathrm{hofib}(K(X) \longrightarrow KSp(X))$

3° $U(X) := \mathrm{hofib}(K(X) \longrightarrow K^h(X))$

KSp = symplectic K-thy.

Now $KQ_m(X) \cong \mathrm{Hom}_{SH(k)}(S^m \wedge X_+, KQ)$

$kq := \tilde{f}_0 KQ$ very effective cover of KQ.

One has

1° $\Sigma^{1,1} kq \xrightarrow{\eta} kq \longrightarrow kgl$.

2° over perfect field of char $\neq 2$, one has

[Ananyevskiy - Röndigs - Østvaer, 17]

$$s_q kq = \begin{cases} \bigvee_{0 \le i \le n} \Sigma^{2n+2i,\, 2n} H\mathbb{Z} & q = 2n \\ \bigvee_{0 \le i \le n} \Sigma^{2n+2i+1,\, 2n+1} H\mathbb{Z} & q = 2n+1 \end{cases}$$

also [Bachmann, 17]

$$vf_q\, kq \simeq \Sigma^{8q,4q} kq$$

§ 2. Computations.

In this section, we try to compute $H\mathbb{Z}$, kgl, kq over $\mathbb{C}$.

Prerequisite: 1) $H^{**}(k; \mathbb{F}_2) = M_2$

$= \mathbb{F}_2[\tau]$ over $\mathbb{C}$

$|\tau| = (0, 1)$

by Voevodsky.

2) $\mathcal{A}$ = Steenrod algebra.

$\mathcal{A}(n)$ = subalgebra gen. by Sq^{2^i}, $i \le n$.

$\mathcal{A} /\!/ \mathcal{A}(n) = \mathcal{A} \otimes_{\mathcal{A}(n)} \mathbb{F}_2$.

e.g. $H\mathbb{F}_2^* H\mathbb{Z} = \mathcal{A} /\!/ \mathcal{A}(0)$

$H\mathbb{F}_2^* ko = \mathcal{A} /\!/ \mathcal{A}(1)$

$H\mathbb{F}_2^* tmf = \mathcal{A} /\!/ \mathcal{A}(2)$

no spectrum E satisfies $H\mathbb{F}_2^* E = \mathcal{A} /\!/ \mathcal{A}(n)$ for $n \ge 3$.

3) mASS

$$E_2 = \mathrm{Ext}_{\mathcal{A}}^{***}(H^{**}(Y;\mathbb{Z}/2),\ H^{**}(X;\mathbb{Z}/2))$$

$$\Rightarrow [X, Y]^{\wedge}_{2,\eta}$$

4) Action of motivic Steenrod alg: for $p=2$

$$Sq^1(\tau) = \rho$$

Cartan formula: $Sq^{2k}(xy) = \sum_{a+b=2k} \tau^{\varepsilon} Sq^a(x) Sq^b(y)$

$$\varepsilon = \begin{cases} 0, & a, b \text{ even} \\ 1, & a, b \text{ odd} \end{cases}$$

$\rho = [-1] \in K_1^M(k)/2$.

1. $H\mathbb{Z}$.

mASS $\Rightarrow$ $E_2 = \mathrm{Ext}_{\mathcal{A}}^{***}(H^{**}H\mathbb{Z}) \Rightarrow (H\mathbb{Z}_{**})^{\wedge}_2$

Since $H^{**}H\mathbb{Z} = \mathcal{A} /\!/ \mathcal{A}(0)$, one get

$$\mathrm{Ext}_{\mathcal{A}}^{***}(H^{**}H\mathbb{Z}) = \mathrm{Ext}_{\mathcal{A}}^{***}(\mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{M}_2).$$

* Change-of-rings formula:

$$\mathrm{Ext}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathcal{B}} M, \mathbb{M}_2) \cong \mathrm{Ext}_{\mathcal{B}}(M, \mathbb{M}_2)$$

for $\mathcal{B} \subseteq \mathcal{A}$ sub-Hopf alg.

$\mathcal{A}$ free right $\mathcal{B}$-mod, true for $\mathcal{B} = \mathcal{A}(n)$, $n \geq 0$.

(can be proved by techinques in cotensors, see green book)

Thus $E_2 = \mathrm{Ext}_{\mathcal{A}(0)}^{***}(\mathbb{M}_2)$

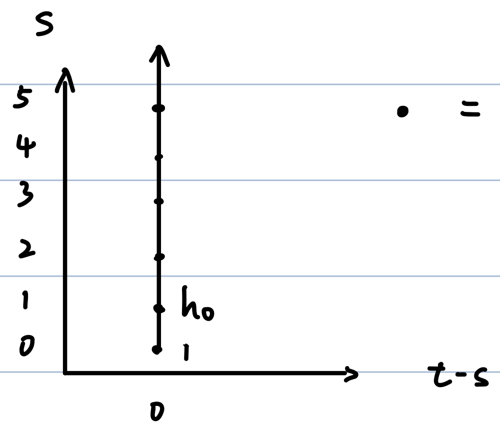
$$= \mathrm{Ext}^{***}_{\mathbb{F}_2[\tau][Sq^1]/(Sq^1)^2}(\mathbb{F}_2[\tau])$$

$$= \mathbb{F}_2[\tau, h_0] = \mathbb{M}_2[h_0]$$

$|h_0| = (1, 1, 0)$

$|\tau| = (0, 0, -1)$

s t w



$\bullet = \mathbb{F}_2[\tau]$.

collapse for deg reason.

So. $(H\mathbb{Z}_{**})^\wedge_2 \simeq \mathbb{Z}_2[\tau]$.

2. kgl.

mASS $\Rightarrow$ $E_2 = \mathrm{Ext}^{***}_A(H^{**}kgl) \Rightarrow (kgl_{**})^\wedge_2$

Note $H^{**}kgl = \mathcal{A} /\!/ E(1)$

$E(1) = \langle Q_0, Q_1 \rangle$

$Q_0 = Sq^1$

$Q_1 = Sq^1Sq^2 + Sq^2Sq^1$.

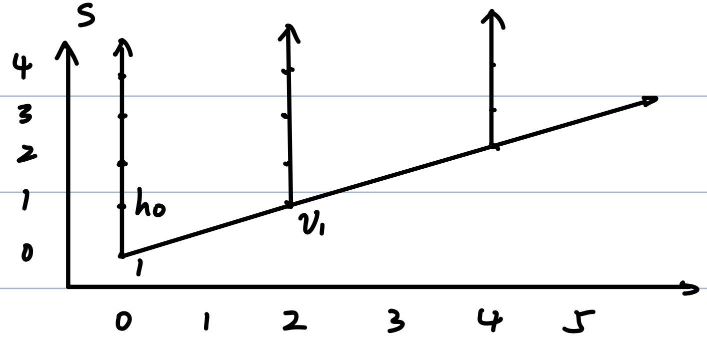
Once again, by change-of-rings.

$$E_2 = \mathrm{Ext}^{***}_A(H^{**}kgl)$$

$$= \mathrm{Ext}^{***}_{E(1)}(\mathbb{M}_2)$$

$E(1)$ is an exterior alg over $\mathbb{M}_2$ on Sq^1, Q_1,

So $E_2 = \mathbb{M}_2[h_0, v_1]$. $|v_1| = (1, 3, 1)$



$\bullet = \mathbb{F}_2[\tau]$

$d_r : E_r^{s,t,u} \longrightarrow E_r^{s+r,\, t-1,\, u}$. d_r respects the wts.

collapse at E_2. So $(kgl_{**})^\wedge_2 \simeq \mathbb{Z}_2[\tau, v_1]$

3. kq.

mASS $\Rightarrow$ $E_2 = \mathrm{Ext}_{\mathcal{A}}^{***}(H^{**} kq)$

$= \mathrm{Ext}_{\mathcal{A}(1)}^{***}(\mathbb{M}_2)$

two ways to compute it:

1) minimal resolution (Isaksen - Shkembi, 10)

2) motivic May SS (Dugger - Isaksen, 09)

$E_1^{****} = \mathbb{M}_2[h_{i,j} : i \geq 1, j \geq 1]$

$\Rightarrow \mathrm{Ext}_{\mathcal{A}}^{***}(\mathbb{M}_2)$

$$\boxed{d_r : E_r^{s,t,u,w} \longrightarrow E_r^{s+1,\, t,\, u-r,\, w}}$$

in our case.

$E_1^{****} = \mathbb{M}_2[h_{i,j} : 1 \leq i+j \leq 2]$

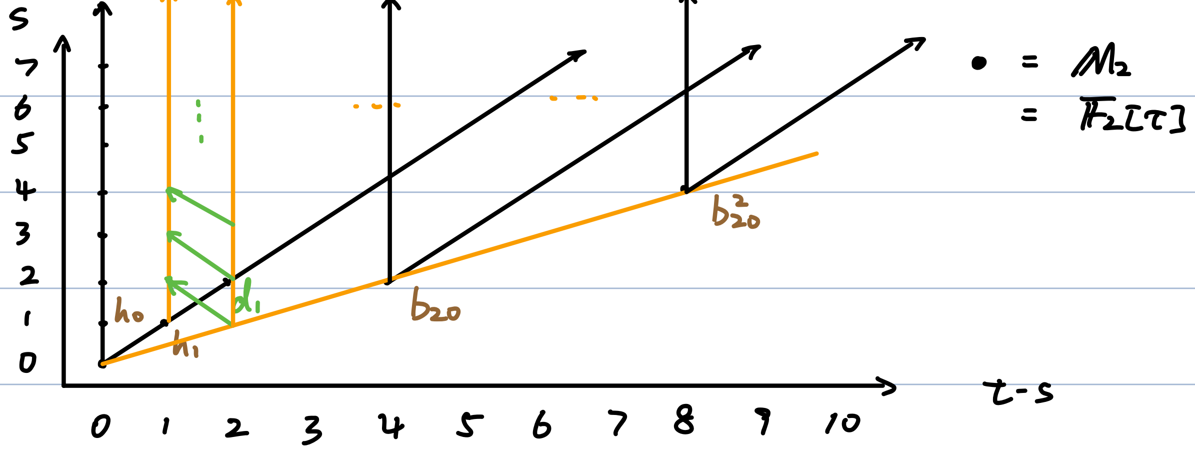
$\Rightarrow \mathrm{Ext}_{\mathcal{A}(1)}^{***}(\mathbb{M}_2)$

induced by $\mathcal{A} \longrightarrow \mathcal{A}(1)$.

where $|h_0| = (1, 1, 0)$

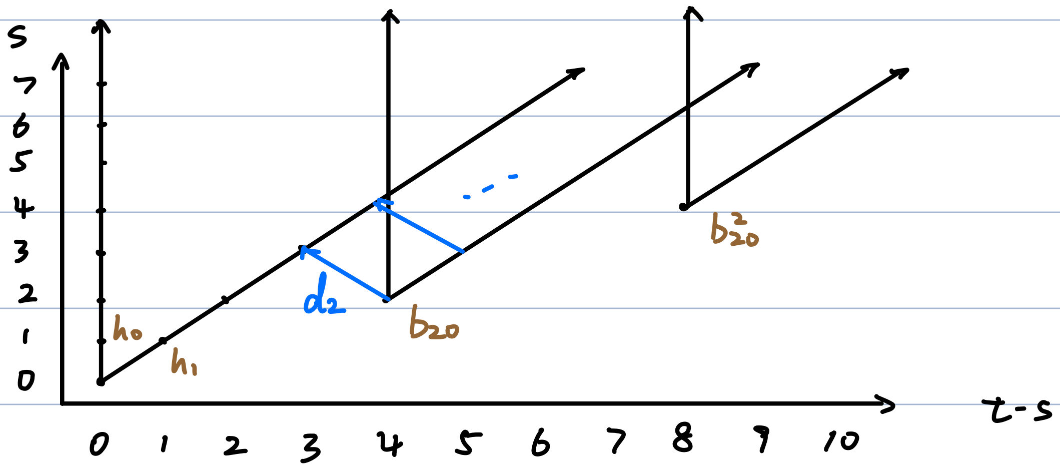
$|h_1| = (1, 2, 1)$

$|h_{20}| = (1, 3, 1)$ $\quad b_{20} = h_{20}^2$



We have $d_1(h_{20}) = h_0 h_1$

$\rightsquigarrow$ E_2 - page



Note that $d_2(b_{20}) = \tau h_1^3$

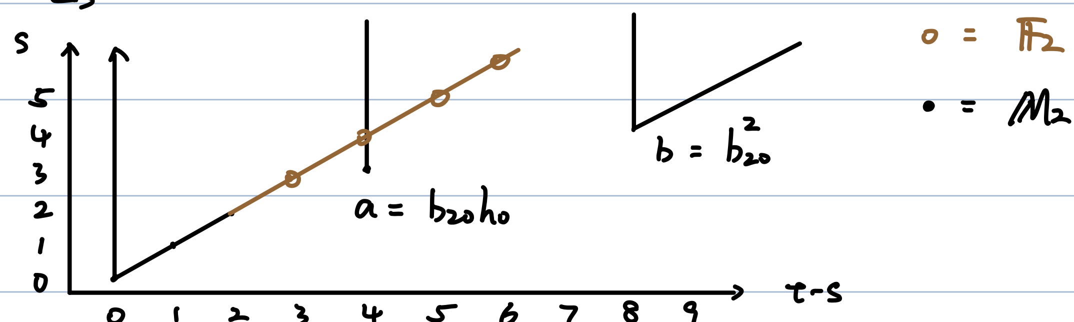
$d_2(\tau^i b_{20}) = \tau^{i+1} h_1^3$ for wt reason

this is by $\langle h_0 , h_1 , h_0 , h_1 \rangle = \{ b_{20} \}$

+ higher Leibniz rule.

[Dugger - Isaksen Table 3 . Section 5]

$\rightsquigarrow$ E_3



$E_3 = E_\infty$ by deg & Massey product reason.

$$\rightsquigarrow \quad (kq_{**})^\wedge_2 \simeq \mathbb{M}_2[h_0, h_1, a, b] \Big/ (h_0h_1, \tau h_1^3, ah_1, a^2 - h_0^2 b)$$

§ 3. Computations over $\mathbb{R}$.

- Key technique : ρ-Bockstein SS [Hill 11].

Note that over $\mathbb{R}$.

$$H^{**}(k; \mathbb{F}_2) = \mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau, \rho]$$

$$|\tau| = (0, 1), \quad |\rho| = (1, 1)$$

ρ-Bockstein : filter ρ-tower. [Hill 11, Culver - Kong - Quigley 21]

Roughly speaking, $\mathrm{Ext}_A^{***}(H_{**}X)$ can be computed by cobar cpx $C_\bullet(\mathbb{M}_2^k, A, H_{**}(X))$. Filter this by powers of ρ.

Namely, $E_1 = \mathrm{Ext}_{A^{\mathbb{C}}}^{***}(\mathbb{M}_2^{\mathbb{C}})[\rho] \qquad d_r(x) = \rho^r y.$

$$\Rightarrow \mathrm{Ext}_{A^{\mathbb{R}}}^{***}(\mathbb{M}_2^{\mathbb{R}})$$

e.g. For $H\mathbb{Z}_{**}^{\mathbb{R}}$, one has

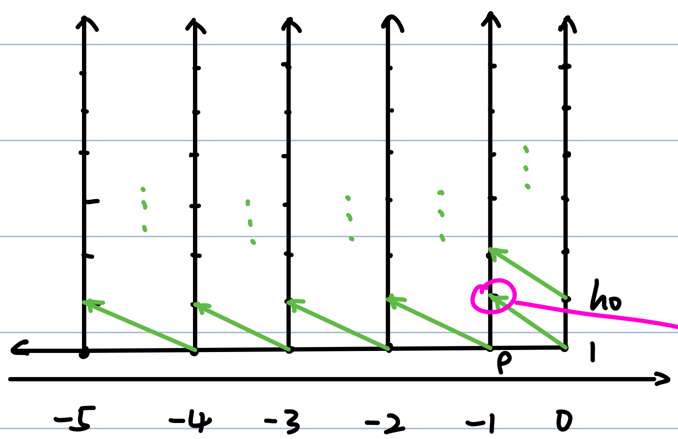
$$E_2 = \mathrm{Ext}_{A(0)^{\mathbb{R}}}^{***}(\mathbb{M}_2^{\mathbb{R}})$$

$$\Uparrow \ \rho\text{-BSS}$$

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})[\rho]$$

$$= \mathbb{M}_2[\rho, h_0]. \qquad |\rho| = (-1,-1)$$

Note that $\begin{cases} d_1(\tau) = \rho h_0 \\ d_1 \text{ linear w.r.t. } h_0,\ \rho. \\ d_1(\tau^2) = 0 \end{cases}$



$\bullet = \mathbb{F}_2[\tau]$

$\mathbb{F}_2\{\cancel{\rho h_0}, \tau\rho h_0, \cancel{\tau^2\rho h_0}, \tau^3\rho h_0, \vdots\}$ τ τ^3

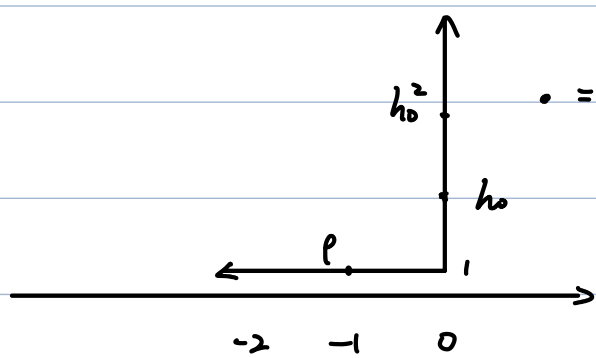
d_1 half kills the dots.

$$d_1(\tau\rho h_0) = d_1(\tau)\cdot\rho h_0 + \tau\cdot\cancel{d_1(\rho h_0)}$$

$$= (\rho h_0)^2$$

by Leibniz rule

$\rightsquigarrow$



$\bullet = \mathbb{F}_2[\tau^2]$

$$\rightsquigarrow \quad (H\mathbb{Z}^{\mathbb{R}}_{**})^\wedge_2 \simeq \mathbb{Z}_2[\tau^2] \oplus \mathbb{F}_2[\tau^2][\rho]\{\rho\}$$

$$\simeq \mathbb{Z}_2[\tau^2, \rho]/(2\rho)$$

<u>Exercise</u> compute $(kgl^{\mathbb{R}}_{**})^\wedge_2$ via ρ-BSS & mASS.

Hint : mASS yields

$$E_2 = \mathrm{Ext}^{***}_{E(1)^{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \Rightarrow (kgl^{\mathbb{R}}_{**})^\wedge_2$$

$$\Uparrow \rho\text{-BSS}$$

$$\mathrm{Ext}^{***}_{E(1)^{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})[\rho]$$

$$= \mathbb{M}_2^{\mathbb{C}}[\rho.h_0.v_1]$$

One has $d_1(\tau) = \rho h_0$, $d_1(\tau^2) = 0$

$d_3(\tau^2) = \rho^3 v_1$. $d_3(\tau^4) = 0$

d_r linear w.r.t. $\rho . h_0 . v_1$.

Use Leibniz rule.

- Betti realization.

If k has a cpx emb.

$$Re : SH(k) \longrightarrow SH$$
$$X \longmapsto X(\mathbb{C})$$

" " real emb.

$$Re : SH(k) \longrightarrow SH^{C_2}$$
$$X \longmapsto X(\mathbb{C})^{C_2 \curvearrowright} \text{ acting by conj.}$$

Both sym. mon. functors !

Application : solving the hidden ext.

$$SH(\mathbb{C}) \longrightarrow SH$$
$$kgl \longmapsto kn$$

$$k_q \longmapsto k_0$$

$$\pi_{i,j} \xmapsto{[\tau^{-1}]} \pi_i$$

$$\mathbb{M}_2 \longmapsto \mathbb{F}_2$$

hidden ext

also hidden ext !