Motivic Cohomology

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Motivation

Recollections: Let X be a finite CW complex and consider $KU^*(X)$

Theorem (Atiyah-Hirzebruch)

There is a spectral sequence

$$E_2^{pq} = H^p(X, KU^q(*)) \implies KU^{p+q}(X)$$

called the Atiyah-Hirzebruch spectral sequence.

Question: does there exist a cohomology theory for schemes which computes the algebraic K-theory?

Motivation

"Motivic cohomology" should be a bigraded functor

$$H^{pq}_M(-, \mathbb{Z}) : Sm/k \to Ab$$

which appears on the E_2-page of a spectral sequence converging to algebraic K-theory (conjectured by Beilinson, Lichtenbaum, Deligne).

Motivation

Let X be smooth.

Conjectures (Beilinson's Dream)

(i) There is an abelian category $\mathcal{MM}_X$ of mixed motivic sheaves on X, and

$$X \mapsto D^b(\mathcal{MM}_X)$$

admits a six-functor formalism.

(ii) $\exists$ Tate sheaves $\mathbb{Z}_X(q) \in \mathcal{MM}_X$ such that

$$H^{pq}_M(X, \mathbb{Z}) \cong H^p(X, \mathbb{Z}_X(q)) \cong Ext^p_{MM_X}(\mathbb{Z}_X(0), \mathbb{Z}_X(q))$$

This is analogous to the case for *singular cohomology* of spaces,

$$H^p(T, \mathbb{Z}) \cong Ext^p_{Shv_T^{Ab}}(\mathbb{Z}_T, \mathbb{Z}_T)$$

Higher Chow Groups

First steps towards motivic cohomology: Bloch's definition of the higher Chow groups of a scheme.

Remark

We may expect some close relationship between Chow groups and K-theory; as a consequence of Grothendieck-Riemann-Roch,

$$ch \otimes \mathbb{Q} : K_0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to CH^*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for X smooth quasi-projective.

Bloch's Cycle Complex

The standard cosimplicial scheme Δ^n is

$$\operatorname{Spec} k[t_0, ..., t_n]/(\sum_i t_i - 1)$$

We let $\partial_i : \Delta^{n-1} \to \{t_i = 0\} \subset \Delta^n$ be the face map. Note $\Delta^n \cong \mathbb{A}^n_k$!

Given a smooth scheme X, we define

$$z^r(X, n) = \mathbb{Z}[Z \subset X \times \Delta^n : \text{codim } r + (*) \ ]$$

Pulling back along

$$\mathrm{id}_X \times \partial_{n,i} : X \times \Delta^{n-1} \to X \times \Delta^n$$

induces face maps

$$\partial^*_{n,i} : z^r(X, n) \to z^r(X, n-1)$$

which maps a subvariety $Z \mapsto X \times \{t_i = 0\} \cap Z$.

Bloch's Cycle Complex

This makes $z^i(X, \bullet)$ into a simplicial abelian group.

We can take alternating sum of face maps $\partial^*_{n,i}$ to obtain boundary map

$$d_n := \sum_i (-1)^i \partial^*_{n,i}$$

and a complex

$$\cdots \to z^r(X, n) \xrightarrow{d_n} z^r(X, n-1) \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} z^r(X, 0) \to 0$$

called the *Bloch Cycle Complex*.

Bloch's Cycle Complex

Definition

The higher Chow groups are defined as

$$CH^r(X,m) := \pi_m(z^r(X,\bullet)) \cong H_m(z^r(X,*))$$

where the above isomorphism is given by the Dold-Kan correspondence.

Higher Chow Groups

Examples

- $CH^i(X,0) \cong CH^i(X)$
- For a field F, $CH^q(\operatorname{Spec} F, m) = 0$ for $m < q$ and

$$CH^q(\operatorname{Spec} F, q) \cong K_q^M(F)$$

Recall Milnor K-theory is defined by

$$K_q^M(F) := (F^\times)^{\otimes q} / \langle (a \otimes (1-a), a \neq 0, 1) \rangle$$

Voevodsky's Definition of Motivic Cohomology

General definition of motivic cohomology provided by Voevodsky, rooted in $\mathbb{A}^1$-homotopy theory. We will need to cover:

- Finite correspondences
- Presheaves with transfers
- Motivic complexes

Finite Correspondences

Fix regular noetherian scheme S/k, with k a perfect field. Let Sm_S be the category of smooth, separated, finite type S-schemes.

Definition

For $X, Y \in Sm_S$, a finite correspondence X to Y is an algebraic cycle

$$\alpha = \sum_i n_i[Z_i] \in X \times_S Y$$

where the irreducible components Z_i are finite and dominant over a connected componets of X.

By the finiteness condition, correspondence is like a multi-valued map $Z \subset X \to Y$.

Finite Correspondences

Example

(i) Let $f : X \to Y$ be a morphism in Sm_S. Then $\Gamma_f \to X \times_S Y$

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_S Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle (f,id)} \\ Y & \xrightarrow[\Delta]{} & Y \times_S Y \end{array}$$

is a closed immersion. Hence morphisms in Sm_S are finite correspondences.
(ii) Assume moreover f finite equidimensional and let

$$\epsilon : X \times_S Y \to Y \times_S X$$

be the map exchanging the factors. Then ${}^t f = e_*[\Gamma_f]$ defines a correspondence from Y to X called the transpose.

Presheaves with Transfers

Let Cor_S be the category whose objects are Sm_S are morphisms given by finite correspondences (correspondences may be composed).

Definition

A presheaf with transfers over S is a functor $F : (Cor_S)^{op} \to Ab$.
A τ-sheaf with transfers additionally required to be a sheaf when restricted to $(Sm_S)_\tau$.

Note for a finite equidimensional morphism $f : X \to Y$ in Sm_S,

$$f_* = F({}^t f) : F(X) \to F(Y)$$

called the transfer.

Effective Motives

Via the Yoneda embedding $Sm_S \hookrightarrow PSh_S^{tr}$

$$X \mapsto c(-, X) =: \mathbb{Z}_S^{tr}(X)$$

Definition

The ∞-category of effective motives is defined to be

$$\mathbf{DM}_S^{\text{eff}} = L_{\mathbb{A}^1}\mathbf{D}\text{Shv}_{Nis}(Cor_S, \mathbb{Z})$$

i.e., Nisnevich sheaves with transfer which are $\mathbb{A}^1$-local.

We will call the class of $\mathbb{Z}_S^{tr}(X)$ in $\mathbf{DM}_S^{\text{eff}}$ the *motive* of X and denote it $M(X)$.

Remark

$M(X)$ generate $\mathbf{DM}_S^{\text{eff}}$ under colimits and suspensions for X/S smooth.

The six functor formalism for $\mathbf{DM}_S^{\text{eff}}$

Let $f : T \to S$ be a morphism of schemes. There is a functor

$$f^* : Cor_S \to Cor_T$$

$$X \mapsto X_T := X \times_S T$$

$$Z \subset X \times_S Y \mapsto Z_T \subset X_T \times_T Y_T$$

and extending linearly on algebraic cycles. If $f : T \to S$ is smooth,

$$f_\sharp : Cor_T \to Cor_S$$

$$X/T \mapsto X/S$$

$$Z \subset X \times_T Y \mapsto \tilde{f}_* Z$$

where $\tilde{f} : X \times_T Y \to X \times_S Y$.

The six functor formalism for $\mathbf{DM}_S^{\text{eff}}$

Letting $f : T \to S$ be a morphism of schemes,

Proposition

The functors $f^*, f_\sharp$ previously defines extend to functors

$$f^* : \mathbf{DM}_S^{\text{eff}} \to \mathbf{DM}_T^{\text{eff}}$$

$$f_\sharp : \mathbf{DM}_T^{\text{eff}} \to \mathbf{DM}_S^{\text{eff}}$$

when defined. By the adjoint functor theorem, both admit right adjoints which we denote $f_*, f^\sharp$, respectively. Moreover there a symmetric monoidal structure $\otimes$ generated by

$$M(X) \otimes M(Y) = M(X \times_S Y)$$

which admits a right adjoint $\underline{\text{Hom}}(-,-)$.

Suslin Complex

We wish to present an alternative formulation of $\mathbb{A}^1$-localization.

If F is a presheaf with transfers over k, let $C_\bullet F$ be the simplicial presheaf given by

$$X \to F(\Delta^\bullet \times_k X)$$

recalling $\Delta^n \cong \mathbb{A}^n$. By taking alternating sums of face maps, we get a complex $C_* F$.

Lemma

For any X, the map $C_* F(X) \to C_* F(X \times_k \mathbb{A}^1)$ is a weak equivalence.

Suslin Complex

Proposition

The Suslin complex functor induces a functor of ∞-categories

$$C_* : D(\mathrm{Sh}^{tr}_{Nis}(S)) \to \mathbf{DM}^{\mathrm{eff}}_S$$

$$F \mapsto \underline{\mathrm{Hom}}(\mathbb{Z}^{tr}_S(\Delta^*), F)$$

which is equivalent to $L_{\mathbb{A}^1}$.

Note that

$$\mathrm{Hom}(\mathbb{Z}^{tr}_k(\Delta^n), F) \cong F(\Delta^n)$$

by the Yoneda lemma since $\mathbb{Z}^{tr}_k(\Delta^n)$ is the presheaf with transfers/k represented by Δ^n.

As a corollary,

$$\boxed{M(X) = C_* \mathbb{Z}^{tr}_S(X)}$$

Motivic Tate Twist

Now we work relative to a perfect field k.

Definition

Voevodsky defines the motivic Tate sheaves by the formula,

$$\mathbb{Z}(n) = C_*(\mathbb{Z}_k^{tr}(\mathbb{G}_m)^{\otimes_{tr}^L n})[-n]$$

(i) $\mathbb{Z}(0) \cong C_*(\mathbb{Z})$, which is quasi-isomorphic to $\mathbb{Z}$ in degree 0

(ii) $\mathbb{Z}(1) \cong M(\mathbb{G}_m)[-1]$

(iii) $\mathbb{Z}(n) \cong \mathbb{Z}(1)^{\otimes n}$ computed in $\mathbf{DM}_k^{\text{eff}}$ $(*)$

Motivic Cohomology

We can now provide Voevodsky's definition of motivic cohomology:

Definition

The motivic cohomology of a smooth k-scheme X in degree (n, i) is

$$H^n(X, \mathbb{Z}(i)) \cong \mathbb{H}^n(X_{Nis}, \mathbb{Z}(i))$$

where we implicitly restrict $\mathbb{Z}(i)$ to X_{Nis}.

As a consequence, we obtain

$$H^n(X, \mathbb{Z}(i)) = \mathrm{Hom}_{\mathbf{DM}_k^{\mathrm{eff}}}(M(X), \mathbb{Z}(i)[n])$$

as conjectured by Beilinson.

Motivic-Chow Comparison

The following is a deep theorem of Voevodsky.

Theorem

Let X be a smooth k-scheme. Then for any indices (n, i),

$$H^n(X, \mathbb{Z}(i)) \cong CH^i(X, 2i - n)$$

An example: we wish to compute

$$\mathrm{Hom}_{\mathbf{DM}_k^{\mathrm{eff}}}(M(X), M(Y))$$

for smooth, proper k-schemes X, Y. Let $d = \dim Y$.

Duality

For a smooth scheme of dimension d, $M^c(T) \cong \underline{\mathrm{Hom}}(M(T), \mathbb{Z}(d)[2d])$

Therefore,

$$\begin{aligned}\mathrm{Hom}(M(X), M(Y)) &\cong \mathrm{Hom}(M(X), \underline{\mathrm{Hom}}(M(Y), \mathbb{Z}(d)[2d])) \\ &\cong \mathrm{Hom}(M(X) \otimes M(Y), \mathbb{Z}(d)[2d]) \\ &\cong \mathrm{Hom}(M(X \times_k Y), \mathbb{Z}(d)[2d]) \\ &\cong CH^d(X \times_k Y)\end{aligned}$$

using $\underline{\mathrm{Hom}}/\otimes$ adjunction, and comparison with Chow groups.

Consequences

Some consequences of motivic-Chow comparison:

Corollary

For a field k, $H^n(\operatorname{Spec} k, \mathbb{Z}(n)) \cong K_n^M(k)$.

Corollary

If X is a smooth k-scheme, $H^n(X, \mathbb{Z}(i)) = 0$ if $2i - n < 0$.

Closely related to one of Beilison's original conjectures for motivic cohomology (which remains open):

Conjecture (Beilinson-Soule Vanishing)

For a smooth k-scheme X, $H^n(X, \mathbb{Z}(i))$ vanishes for all $n < 0$.

Returning to our original point,

Theorem (Bloch-Lichtenbaum, Friedlander-Suslin)

For a smooth scheme X, there is a spectral sequence

$$E_2^{n,i} = H^{n-i}(X, \mathbb{Z}(-q)) \implies K_{-p-q}(X)$$

whose E_2-page is Motivic cohomology, which converges to the algebraic K-theory of X.

Thomason constructs a spectral sequence, which goes from the ℓ-adic etale cohomology to ℓ-adic K-theory (with Bockstein inverted),

$$E_2^{pq} = \begin{cases} H^p_{et}(X, \mathbb{Z}/\ell(i)), & q = 2i \\ 0 \end{cases} \implies K/\ell_{q-p}(X)[\beta^{-1}]$$

This should be subsumed by the above result, though I am unsure of details.

Some additional slides (taken from Marc Levine):

Recall,

- A $\mathbb{P}^1$-spectrum $E \in \mathrm{SH}(k)$ defines a bigraded cohomology theory on Sm_k by
$$E^{p,q}(X) := [\Sigma^\infty_{\mathbb{P}^1} X_+, \Sigma^{p,q} X]_{\mathrm{SH}(k)}$$
- E has bigraded homotopy sheaves: sheafify the presheaf
$$X \mapsto [\Sigma^{p,q}\Sigma^\infty_{\mathbb{P}^1} X_+, E]_{\mathrm{SH}(k)} = E^{-p,-q}$$

Voevodsky constructed a $\mathbb{P}^1$-spectrum $H\mathbb{Z}$ in $\mathrm{SH}(k)$ representing motivic cohomology and a $\mathbb{P}^1$-spectrum KGL represented algebraic K-theory;

$$H\mathbb{Z}^{p,q}(X) = H^p(X, \mathbb{Z}(q))$$
$$KGL^{p,q} = K_{2q-p}(X)$$

The classical Moore-Postnikov tower in SH filters a spectrum E by

$$\cdots \to \tau_{\geq n+1}E \to \tau_{\geq n}E \to \cdots \to E$$

where $\tau_{\geq n}E$ is the $(n-1)$-connected cover of E.

Voevodsky defined a filtration on SH(k) by "$\mathbb{P}^1$-connectivity", yielding the *slice tower*,

$$\cdots \to f_{n+1}E \to f_nE \to \cdots \to E$$

where f_nE is the $\mathbb{P}^1$-$(n-1)$ connected cover of E.

The layers $s_nE := f_nE/f_{n+1}E$ is called the nth slice of E.

Theorem (Vovodeksy, Levine, Bachmann-Elmanto)

$s_0(\mathbb{S}_k) = H\mathbb{Z}$, $s_n(KGL) = \Sigma^n_{\mathbb{P}^1} H\mathbb{Z}$

Moreover, the spectral sequence arising from the slice tower for KGL recovers the Bloch-Lichtenbaum/Friedlander-Suslin spectral sequence (*which I stated incorrectly in lecture*).

References

[Lev] Motivic cohomology, past, present and future (IMU lecture)

[Voe02] A possible new approach to the motivic spectral sequence for algebraic K-theory

[MVW06] Lecture Notes on Motivic Cohomology

[Dég21] An introductory course on Voevodsky's motivic complexes