導来圏 et 導来函手 en Géométrie Algébrique

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Contents

References

Expository notes:

- Schapira, *Categories and Homological Algebra.* [Schapira]
- Bridgeland, *D^b (Intro).*
- Căldăraru, *Derived Categories of Sheaves: A Skimming.*

Books:

- Huybrechts, *Fourier–Mukai Transforms in Algebraic Geometry.*
- Hartshorne, *Residues and Duality.*
- 李文威，代数学方法 II (未定稿). [李文威]
- Weibel, *An Introduction to Homological Algebra* (Chap. 10 on derived categories).
- Bocklandt, *A Gentle Introduction to Homological Mirror Symmetry* (Chap. 7 on the B-side).

Overview

Kontsevich's homological mirror symmetry is a vague conjecture about the derived equivalence of the A_∞-categories

$$\mathsf{D}^\pi\mathsf{Fuk}(X) \simeq \mathsf{D}^\mathrm{b}\mathsf{Coh}(X^\vee)$$

for a mirror pair $(X, X^\vee)$ of Calabi–Yau varieties. The left-hand side is the derived Fukaya category constructed from the symplectic geometry of X, known as the A-model, whereas the right-hand side is the bounded derived category of coherent sheaves on $X^\vee$, known as the B-model. These notes aim to fill in the gaps between undergraduate algebraic geometry and the essential backgrounds of understanding $\mathsf{D}^\mathrm{b}\mathsf{Coh}(X)$ when X is a smooth projective variety.

Some topics and results in derived categories of sheaves to be covered:

- Some initial results, e.g. $\mathsf{D}^\mathrm{b}\mathsf{Coh}(X) \cong \mathsf{D}^\mathrm{b}_{\mathsf{Coh}}(\mathsf{QCoh}(X))$;
- $\mathsf{D}^\mathrm{b}\mathsf{Coh}(\mathbb{P}^1) \cong \mathsf{D}\,\mathsf{Rep}\,Q$ for the Kronecker quiver Q;
- Derived category of projective n-spaces $\mathsf{D}^\mathrm{b}\mathsf{Coh}(\mathbb{P}^n) = \langle \mathcal{O}(-n), ..., \mathcal{O}(-1), \mathcal{O}(0)\rangle$;
- Smoothness, perfect complexes, $\mathsf{Perf}\,X = \mathsf{D}^\mathrm{b}_{\mathsf{Coh}}(X)$ for regular Noetherian scheme X;
- Serre functor, derived Serre duality;

- Grothendieck–Verdier duality;
- Ampleness, canonical bundle, Fano & Calabi–Yau varieties;
- **Bondal–Orlov Theorem.** Suppose that X is a projective variety with canonical bundle ω_X ample or anti-ample, and Y is a projective variety. If $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \cong \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$ as triangulated categories, then $X \cong Y$ as varieties;
- A_∞-structure on $\mathsf{Coh}(X)$.

I will continue from the notes ([YS]) *Triangulated categories and derived categories* by Jinghui Yang & Shuwei Wang. **Warning.** Currently these notes grew out from a talk and was not self-contained in nature. In the future they may be extended to a more inclusive version, where I aim to present derived categories and localisations rigourously.

0 Derived Functors

This section mainly follows [李文威]. The relevant sections are 1.8, 1.11, 3.2, 4.6–4.9, 4.12.

Recall that from an Abelian category $\mathcal{A}$ we can build the **homotopy category** $\mathsf{K}(\mathcal{A})$ by taking quotient by chain maps homotopic to zero in the chain complex category $\mathsf{Ch}(\mathcal{A})$, and the **derived category** $\mathsf{D}(\mathcal{A})$ by (Verdier) localisation on the acyclic complexes in $\mathsf{K}(\mathcal{A})$. In particular, every quasi-isomorphism of chains in $\mathcal{A}$ becomes an isomorphism in $\mathsf{D}(\mathcal{A})$ (and $\mathsf{D}(\mathcal{A})$ is universal with respect to this property by construction). In general, $\mathsf{K}(\mathcal{A})$ and $\mathsf{D}(\mathcal{A})$ are not Abelian, but rather **triangulated categories**. For all the technical details we refer to the notes from the previous talk. If $\mathcal{A}$ has enough injectives, then $\mathsf{D}^+(\mathcal{A})$ is equivalent to $\mathcal{I}_{\mathcal{A}}$, the full subcategory of injective objects of $\mathcal{A}$.

There is a natural way to define derived functor under the viewpoint of derived categories. First we recall the classical definition. Suppose that $\mathcal{A}$ is an Abelian category with enough injectives. For $A \in \mathrm{Obj}(\mathcal{A})$, let $A \to I^\bullet$ be an injective resolution of A. Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor. Then the ***n*-th right derived functor** of F acting on X is given by $\mathsf{R}^n F(A) := \mathrm{H}^n(F(I^\bullet))$.

Let $\mathcal{K}$ and $\mathcal{K}'$ be triangulated categories, and $Q\colon \mathcal{K} \to \mathcal{K}/\mathcal{N}$ and $Q'\colon \mathcal{K}' \to \mathcal{K}'/\mathcal{N}'$ be Verdier localisations. Suppose that $F\colon \mathcal{K} \to \mathcal{K}'$ is a triangulated functor (i.e. preserving distinguished triangles). The naive idea is to seek for a functor G such that the following diagram commutes (and satisfies some universal properties):

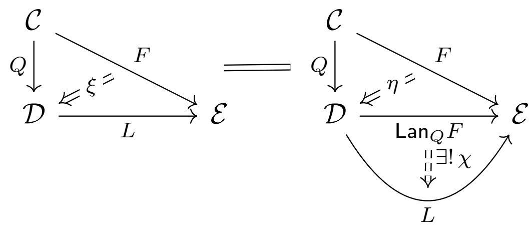
$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\;F\;} & \mathcal{K}' \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q'} \\ \mathcal{K}/\mathcal{N} & \overset{G}{\dashrightarrow} & \mathcal{K}'/\mathcal{N}' \end{array}$$

For this we need the Kan extension from category theory. Let's recap.

Definition 0.1. Consider functors $Q\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{E}$. The **left Kan extension** of F by Q consists of the following data:

- A functor $\mathsf{Lan}_Q F\colon \mathcal{D} \to \mathcal{E}$;
- A natural transformation $\eta\colon F \Rightarrow \mathsf{Lan}_Q F \circ Q$;

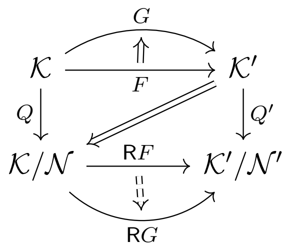
which satisfy the following universal property: for any functor $L\colon \mathcal{D} \to \mathcal{E}$ and natural transformation $\xi\colon F \Rightarrow L \circ Q$, there exists a unique $\chi\colon \mathsf{Lan}_Q F \Rightarrow L$ such that $\xi = (\chi \circ Q) \circ \eta$.



Considering left Kan extension in the opposite categories, we could define **right Kan extension**. The corresponding diagram is given by reversing all natural transformations in the above diagram.

Definition 0.2. Let $F\colon \mathcal{K} \to \mathcal{K}'$ as above. If the left (*resp.* right) Kan extension $\mathsf{Lan}_Q(Q' \circ F)$ (*resp.* $\mathsf{Ran}_Q(Q' \circ F)$) exists and is a triangulated functor, then it is called the right (*resp.* left) **derived functor** of F, denoted by $\mathsf{R}F$ (*resp.* $\mathsf{L}F$).

Remark. Suppose that $G\colon \mathcal{K} \to \mathcal{K}'$ is another triangulated functor with a natural transformation $\eta\colon F \Rightarrow G$. If the right derived functor $\mathsf{R}G$ exists, then there is a canonical natural transformation $\mathsf{R}F \Rightarrow \mathsf{R}G$ by the universal property of right Kan extension.



Then we focus on the derived categories. Note that an additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ between Abelian categories induces the homotopy functor $\mathsf{K}F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}')$[1] which is triangulated. Consider the Kan extensions:

Assuming existence, $\mathsf{R}F$ (*resp.* $\mathsf{L}F$) is called the right (*resp.* left) derived functor of F. Their uniqueness is ensured by the universal property. What about existence?

Definition 0.3. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be as above. Let $\mathcal{J}$ be a triangulated subcategory of $\mathsf{K}(\mathcal{A})$. We say that $\mathcal{J}$ is ***F*-injective** (*resp.* ***F*-projective**), if:

- Resolution: For $X \in \mathrm{Obj}(\mathsf{Ch}(\mathcal{A}))$ there exists $Y \in \mathrm{Obj}(\mathcal{J})$ and a quasi-isomorphism $X \to Y$ (*resp.* $Y \to X$).
- Preserving null system: $F(\mathrm{Obj}(\mathcal{N}(\mathcal{A}) \cap \mathcal{J})) \subseteq \mathrm{Obj}(\mathcal{N}(\mathcal{A}'))$

Note that here the null system $\mathcal{N}(\mathcal{A})$ is the acyclic complexes in $\mathsf{Ch}(\mathcal{A})$.

[1] The cases for K^+, K^-, and K^{b} are identical.

Remark. There is a similar notion for subcategories of $\mathcal{A}$. Let $\mathcal{I}$ be an additive full subcategory of $\mathcal{A}$. We say that $\mathcal{I}$ is of **type I** (*resp.* **type P**) relative to F, if:

- For any $X \in \mathrm{Obj}(\mathcal{A})$ there exists $Y \in \mathrm{Obj}(\mathcal{I})$ and a monomorphism $X \to Y$ (*resp.* epimorphism $Y \to X$);
- For any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, if $X, Y \in \mathrm{Obj}(\mathcal{I})$ then $Z \in \mathrm{Obj}(\mathcal{I})$. (*resp.* If $Y, Z \in \mathrm{Obj}(\mathcal{I})$ then $X \in \mathrm{Obj}(\mathcal{I})$.) In this case $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is also exact.

This should be considered as the generalisation of injective objects in $\mathcal{A}$. Indeed the subcategory $\mathcal{I}_{\mathcal{A}}$ of injective objects of $\mathcal{A}$ is of type I relative to any additive functor F.

The terminology is taken from [李文威, 4.8.2]. In fact, this notion is what [Schapira, 4.7.5] calls F-*injective*. The two definitions are closely related. If $\mathcal{I} \subseteq \mathcal{A}$ is of type I relative to F, then $\mathsf{K}(\mathcal{I}) \subseteq \mathsf{K}(\mathcal{A})$ is F-injective.

Proposition 0.4

Let $F\colon \mathcal{A} \to \mathcal{A}'$ be as above. Suppose that $\mathsf{K}(\mathcal{A})$ has an F-injective (*resp.* F-projective) subcategory. Then the right (*resp.* left) derived functor $\mathsf{R}F$ (*resp.* $\mathsf{L}F$) exists.

Proof. Let $\mathcal{I}$ be an F-injective subcategory of $\mathsf{K}(\mathcal{A})$. By Theorem 3.5 in [YS], there is an equivalence of category $\mathsf{D}(\mathcal{A}) \simeq \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I})$. Since $F(\mathrm{Obj}(\mathcal{N}(\mathcal{A}) \cap \mathcal{I})) \subseteq \mathrm{Obj}(\mathcal{N}(\mathcal{A}'))$, by the universal property of Verdier localisation there is a functor $F^\flat\colon \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I}) \to \mathsf{D}(\mathcal{A}')$. Take $\mathsf{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$ to be the functor such that the following diagram commutes:

$$\begin{array}{ccc} \mathsf{D}(\mathcal{A}) & \xrightarrow{\mathsf{R}F} & \mathsf{D}(\mathcal{A}') \\ {\scriptstyle i^{-1}}\big\uparrow\big\downarrow{\scriptstyle i} & \nearrow_{F^\flat} & \\ \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I}) & & \end{array}$$

Next we need to verify that $\mathsf{R}F$ is indeed the Kan extension. See [李文威, Prop 1.11.2, Prop 4.6.4]. □

Corollary 0.5

Suppose that $\mathcal{A}$ has enough injectives (*resp.* projectives). Then the right (*resp.* left) derived functor ${}^{+}\mathsf{R}F$ (*resp.* ${}^{+}\mathsf{L}F$) exists for any additive functor $F\colon \mathcal{A} \to \mathcal{A}'$.

Proof. Immediate by [YS, Prop 3.10]. □

Proposition 0.6

Suppose that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be a left exact additive functor. Then for $A \in \mathrm{Obj}(\mathcal{A})$. we have

$$\mathsf{R}^n F(A) = \mathrm{H}^n \circ \mathsf{R}F(QA),$$

where $QA \in \mathsf{D}^{+}(\mathcal{A})$ and $\mathrm{H}^n\colon \mathsf{D}^{+}(\mathcal{A}') \to \mathsf{Ab}$ is the n-th cohomology functor.

Proof. Take an injective resolution $A \to I^\bullet$. This gives rise to a quasi-isomorphism $A \to I$ in $\mathsf{K}^+(\mathcal{A})$, where I lies in the F-injective subcategory $\mathsf{K}^+(\mathcal{I}_\mathcal{A})$ of $\mathsf{K}^+(\mathcal{A})$. Now we have the isomorphisms

$$\mathsf{R}F(QA) \cong \mathsf{R}F(QI) \cong Q'\mathsf{K}^+F(I).$$

Applying H^n gives the result. □

Proposition 0.7. Long Exact Sequence

Suppose that $F\colon \mathcal{A} \to \mathcal{A}'$ has a right derived functor $\mathsf{R}F$. For any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathsf{D}(\mathcal{A})$, there is a canonical long exact sequence:

$$\cdots \longrightarrow \mathsf{R}^{n-1}(Z) \longrightarrow \mathsf{R}^n F(X) \longrightarrow \mathsf{R}^n F(Y) \longrightarrow \mathsf{R}^n F(Z) \longrightarrow \mathsf{R}^{n+1} F(X) \longrightarrow \cdots$$

Proof. Since $\mathsf{R}F$ is a triangulated functor, the result follows from applying the cohomology functor H^0. □

Comparing to the classical definition, a great advantage of derived functors in this viewpoint is that they compose in a canonical way.

Proposition 0.8

Consider the additive functors among Abelian categories:

$$\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}''$$

Suppose that the right derived functors $\mathsf{R}F$, $\mathsf{R}F'$ and $\mathsf{R}(F' \circ F)$ all exist. Then there is a natural transformation $\mathsf{R}(F' \circ F) \Rightarrow (\mathsf{R}F') \circ (\mathsf{R}F)$.

Moreover, if $\mathcal{I}$ is an F-injective subcategory of $\mathsf{K}(\mathcal{A})$ and $\mathcal{I}'$ is an F'-injective subcategory of $\mathsf{K}(\mathcal{A}')$ such that $F(\mathrm{Obj}(\mathcal{I})) \subseteq \mathrm{Obj}(\mathcal{I}')$, then $\mathcal{I}$ is $F' \circ F$-injective. And the natural transformation above is an isomorphism:

$$\mathsf{R}(F' \circ F) \cong (\mathsf{R}F') \circ (\mathsf{R}F).$$

Proof. For the first part, the natural transformation $\mathsf{R}(F' \circ F) \Rightarrow (\mathsf{R}F') \circ (\mathsf{R}F)$ is induced by the universal property of left Kan extensions (*check it!*) For the second part, take $I \in \mathrm{Obj}(\mathcal{I})$. Using the construction in Proposition 0.4 we obtain

$$(\mathsf{R}F') \circ (\mathsf{R}F)(QI) = Q'' \circ F' \circ F(I) = \mathsf{R}(F' \circ F)(QI)$$

For $X \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}))$, by choosing quasi-isomorphism $X \to I$ we obtain the isomorphism $(\mathsf{R}F') \circ (\mathsf{R}F)(QX) \cong \mathsf{R}(F' \circ F)(QX)$. Finally check that this is compatible with the natural transformation given above. □

Derived Bi-Functors

The tensor functor $- \otimes -$ and the Hom functor $\mathrm{Hom}(-,-)$ are two typical examples of bi-functors of Abelian categories. Since we are interested in these functors, it is useful to treat the derived bi-functors separately.

Definition 0.9. Let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ be triangulated categories. A bi-functor $F\colon \mathcal{K}_1 \times \mathcal{K}_2 \to \mathcal{K}$ is triangulated, if

- F is triangulated in both slots;
- For any $A \in \mathcal{K}_1$ and $B \in \mathcal{K}_2$, the following diagram anti-commutes[2]:

$$\begin{array}{ccc} F(\mathsf{T}_1 A, \mathsf{T}_2 B) & \longrightarrow & \mathsf{T}F(A, \mathsf{T}_2 B) \\ \downarrow & & \downarrow \\ \mathsf{T}F(\mathsf{T}_1 A, B) & \longrightarrow & \mathsf{T}^2 F(A, B) \end{array}$$

The definition of the left/right derived functor of a triangulated bi-functor is essentially identical. We are interested in the cases where the triangulated categories are homotopy categories of Abelian categories.

Now we consider Abelian categories $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$, where $\mathcal{A}$ admits countable products and coproducts. Let $F\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive bi-functor. Let

$$\begin{aligned} \mathsf{Ch}_{\oplus} F &:= \mathrm{Tot}_{\oplus} \circ \mathsf{Ch}^2(F) \colon \mathsf{Ch}(\mathcal{A}_1) \times \mathsf{Ch}(\mathcal{A}_2) \to \mathsf{Ch}(\mathcal{A}); \\ \mathsf{Ch}_{\Pi} F &:= \mathrm{Tot}_{\Pi} \circ \mathsf{Ch}^2(F) \colon \mathsf{Ch}(\mathcal{A}_1) \times \mathsf{Ch}(\mathcal{A}_2) \to \mathsf{Ch}(\mathcal{A}). \end{aligned}$$

Then induce the triangulated bi-functors $\mathsf{K}_{\oplus} F, \mathsf{K}_{\Pi} F\colon \mathsf{K}(\mathcal{A}_1) \times \mathsf{K}(\mathcal{A}_2) \to \mathsf{K}(\mathcal{A})$.

Let $\mathcal{I}_1, \mathcal{I}_2$ be triangulated subcategories of $\mathsf{K}(\mathcal{A}_1), \mathsf{K}(\mathcal{A}_2)$ respectively. We say that $(\mathcal{I}_1, \mathcal{I}_2)$ is F-injective (*resp.* F-projective), if $\mathcal{I}_2$ is $F(A_1, -)$-injective for any $A_1 \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}_1))$, and $\mathcal{I}_1$ is $F(-, A_2)$-injective for any $A_2 \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}_2))$.

Proposition 0.10

Let $F\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be as above.

1. If $(\mathcal{I}_1, \mathcal{I}_2)$ is F-injective, then $\mathsf{R}F := \mathsf{R}\mathsf{K}_{\Pi} F$ exists. We call it the right derived functor of F;
2. If $(\mathcal{P}_1, \mathcal{P}_2)$ is F-projective, then $\mathsf{L}F := \mathsf{L}\mathsf{K}_{\oplus} F$ exists. We call it the left derived functor of F.

Ext and $\mathsf{R}\,\mathrm{Hom}$

Recall that in *C2.2 Homological Algebra.* we define the $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ to be the n-th right derived functor of $\mathrm{Hom}_{\mathcal{A}}(A, -)$ acting on $B \in \mathrm{Obj}(\mathcal{A})$. If $\mathcal{A}$ has enough injectives or projectives, then $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ is computed by an injective resolution $B \to I^\bullet$ of B or a projective resolution $P^\bullet \to A$ of A. By acyclic assembly lemma, $\mathrm{Ext}^n_{\mathcal{A}}(A, B)$ can also be computed as the n-th cohomology of the total complex $\mathrm{Tot}^{\Pi}(\mathrm{Hom}_{\mathcal{A}}(P_\bullet, Q_\bullet))$ using projective resolutions $P_\bullet \to A$ and $Q_\bullet \to B$.

Using the derived category, the Ext group can be defined without using injective or projective resolutions:

Definition 0.11. Let $\mathcal{A}$ be an Abelian category. For chain complexes A, B in $\mathsf{Ch}(\mathcal{A})$, we define the **(hyper-)Ext** group as

$$\mathrm{Ext}^n_{\mathcal{A}}(A, B) := \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]).$$

[2]The term is used in [李文威]. It means that the two composite morphisms in the square differ by a sign.

This definition gives an obvious multiplication structure on Ext:

$$\begin{aligned} \mathrm{Ext}^n_{\mathcal{A}}(B,C) \times \mathrm{Ext}^m_{\mathcal{A}}(A,B) &\longrightarrow \mathrm{Ext}^{n+m}_{\mathcal{A}}(A,C) \\ (f,g) &\longmapsto f[m]\circ g \end{aligned}$$

In particular it makes $\mathrm{Ext}^\bullet_{\mathcal{A}}(A,A)$ a graded ring for any $A \in \mathrm{Obj}(\mathcal{A})$.

Next we will consider Ext as the right derived functor of Hom bi-functor $\mathrm{Hom}_{\mathcal{A}} \colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$. It induces the functor on the double complexes:

$$\mathrm{Hom}^{\bullet;\bullet}_{\mathcal{A}}(-,-) \colon \mathsf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Ab}) \times \mathsf{Ch}(\mathsf{Ab}).$$

Define $\mathsf{Ch}\,\mathrm{Hom}_{\mathcal{A}}(-,-) := \mathrm{Tot}_{\Pi}\,\mathrm{Hom}^{\bullet;\bullet}_{\mathcal{A}}(-,-) \colon \mathsf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Ab})$. It is not hard to verify that $\mathsf{Ch}\,\mathrm{Hom}_{\mathcal{A}}$ is naturally isomorphic to the **Hom complex** $\mathrm{Hom}^\bullet_{\mathcal{A}}$:

$$\mathrm{Hom}^n_{\mathcal{A}}(A,B) := \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(A^k, B^{k+n}), \qquad \mathrm{d}^n_{\mathrm{Hom}}(f) := \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A.$$

Lemma 0.12

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A, B[n]) \cong \mathrm{H}^n(\mathrm{Hom}^\bullet_{\mathcal{A}}(A,B), \mathrm{d}^\bullet_{\mathrm{Hom}}).$$

Proof. Trivial by definition. □

The bi-functor $\mathsf{Ch}\,\mathrm{Hom}_{\mathcal{A}}$ or $\mathrm{Hom}^\bullet_{\mathcal{A}}$ induces the triangulated bi-functor

$$\mathsf{K}\,\mathrm{Hom}_{\mathcal{A}} \colon \mathsf{K}^-(\mathcal{A})^{\mathrm{op}} \times \mathsf{K}^+(\mathcal{A}) \to \mathsf{K}^+(\mathsf{Ab}).$$

If $\mathcal{A}$ has enough injectives or projectives, then the right derived functor

$$\mathsf{R}\,\mathrm{Hom}_{\mathcal{A}} \colon \mathsf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}^+(\mathsf{Ab})$$

exists.

Proposition 0.13

Suppose that $\mathcal{A}$ has enough injectives or projectives. For $A \in \mathrm{Obj}(\mathsf{D}^-(\mathcal{A}))$ and $B \in \mathrm{Obj}(\mathsf{D}^+(\mathcal{A}))$, there exists a canonical isomorphism

$$\mathrm{H}^n\,\mathsf{R}\,\mathrm{Hom}_{\mathcal{A}}(A,B) \cong \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]).$$

Proof. Taking the right derived functor in the previous lemma and note that the cohomology functor H^n factors through the derived functor. □

Corollary 0.14

Suppose that $\mathcal{A}$ has enough injectives. Let $A, B \in \mathrm{Obj}(\mathcal{A})$ (viewed as complexes concentrated at degree 0). Then there is a canonical isomorphism

$$\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]) \cong \mathsf{R}^n \mathrm{Hom}(A, -)(B)$$

Therefore the hyper-Ext is a generalisation of the usual Ext.

Tor and $\otimes^{\mathsf{L}}$

In this part we only consider R-modules. For $A, B \in \mathsf{Ch}(R\text{-}\mathsf{Mod})$, from *C3.1 Algebraic Topology* we recall the tensor product of complexes is given by the total complex $A \otimes_R B := \mathrm{Tot}_{\oplus}(A^\bullet \otimes_R B^\bullet)$.

Definition 0.15. For $A, B \in \mathsf{Ch}(R\text{-}\mathsf{Mod})$, the **total tensor product** of A and B is the left derived functor

$$A \otimes_R^{\mathsf{L}} B := \mathsf{L}(- \otimes_R -)(A, B).$$

$\mathsf{L}(- \otimes_R -)\colon \mathsf{D}^-(\mathsf{Mod}\text{-}R) \times \mathsf{D}^-(R\text{-}\mathsf{Mod}) \to \mathsf{D}^-(\mathsf{Ab})$ exists because R-Mod has enough projectives. By taking cohomology we have the **(hyper-)Tor** groups:

$$\mathrm{Tor}_n^R(A, B) := \mathrm{H}_n(A \otimes_R^{\mathsf{L}} B).^3$$

Similar as hyper-Ext, using the theory of derived functors we can verify that the hyper-Tor reduces to the usual Tor on $\mathrm{Obj}(R\text{-}\mathsf{Mod})$ (defined using projective resolutions).

Remark. In general $\mathsf{QCoh}(X)$ does not have enough projectives. We will have to instead use flat resolutions to compute the total tensor product. See later.

Proposition 0.16. Derived Tensor-Hom Adjunction

Let $A \in \mathsf{D}(\mathsf{Mod}\text{-}R)$, $B \in \mathsf{D}(R\text{-}\mathsf{Mod})$, and $C \in \mathsf{D}(\mathsf{Ab})$. There are canonical isomorphisms in $\mathsf{D}(\mathsf{Ab})$:

$$\begin{aligned}\mathsf{R}\,\mathrm{Hom}_{\mathsf{Ab}}(X \otimes_R^{\mathsf{L}} Y, Z) &\cong \mathsf{R}\,\mathrm{Hom}_{\mathsf{Mod}\text{-}R}(X, \mathsf{R}\,\mathrm{Hom}_{\mathsf{Ab}}(Y, Z)) \\ &\cong \mathsf{R}\,\mathrm{Hom}_{R\text{-}\mathsf{Mod}}(Y, \mathsf{R}\,\mathrm{Hom}_{\mathsf{Ab}}(X, Z)).\end{aligned}$$

[3] Cohomology and homology make no difference in algebra. By convention, $\mathrm{H}_n := \mathrm{H}^{-n}$.