導来圏 et 導来函手 en Géométrie Algébrique

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Contents

References

Expository notes:

- Jinghui Yang & Shuwei Wang, *Triangulated categories and derived categories* [YS]
- Schapira, *Categories and Homological Algebra.* [Scha]
- Bridgeland, *D^b (Intro).*
- Căldăraru, *Derived Categories of Sheaves: A Skimming.* [Căld]
- Calabrese, *On a Theorem of Beĭlinson.*

Books:

- Huybrechts, *Fourier–Mukai Transforms in Algebraic Geometry.* [Huyb]
- Hartshorne, *Algebraic Geometry.* [HartsAG]
- Hartshorne, *Residues and Duality.*
- 李文威，代数学方法 II (未定稿). [李文威]
- Bocklandt, *A Gentle Introduction to Homological Mirror Symmetry* (Chap. 7 on the B-side).

Prerequisites (Oxford courses):

- B2.2 Commutative Algebra
- C2.2 Homological Algebra
- C2.6 Introduction to Schemes
- C3.1 Algebraic Topology
- C3.4 Algebraic Geometry

I will take everything from those courses for granted.

Overview

Kontsevich's homological mirror symmetry is a conjecture on the derived equivalence of the A_∞-categories

$$\mathsf{D}^\pi\mathsf{Fuk}(X) \simeq \mathsf{D}^\mathrm{b}\mathsf{Coh}(X^\vee)$$

for a mirror pair $(X, X^\vee)$ of Calabi–Yau varieties. The left-hand side is the derived Fukaya category constructed from the symplectic geometry of X, known as the A-model, whereas the right-hand side is the bounded derived category of coherent sheaves on $X^\vee$, known as the B-model. These notes aim to fill in the gaps between undergraduate algebraic geometry and the essential backgrounds of understanding $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X)$ when X is a smooth projective variety.

Some topics and results in derived categories of sheaves to be covered:

- Some initial results, e.g. $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \cong \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(\mathsf{QCoh}(X))$;
- Smoothness, perfect complexes, $\mathsf{Perf}\, X = \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ for regular Noetherian scheme X;
- Serre functor, derived Serre duality;
- Grothendieck–Verdier duality;
- Ampleness, canonical bundle, Fano & Calabi–Yau varieties;
- **Bondal–Orlov Theorem.** Suppose that X is a projective variety with canonical bundle ω_X ample or anti-ample, and Y is a projective variety. If $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \cong \mathsf{D}^{\mathrm{b}}\mathsf{Coh}(Y)$ as triangulated categories, then $X \cong Y$ as varieties;
- A_∞-structure on $\mathsf{Coh}(X)$;
- $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(\mathbb{P}^1) \cong \mathsf{D}\,\mathsf{Rep}\, Q$ for the Kronecker quiver Q;
- Derived category of projective n-spaces $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(\mathbb{P}^n) = \langle \mathcal{O}(-n), ..., \mathcal{O}(-1), \mathcal{O}(0)\rangle$;

I will continue from the notes ([YS]) *Triangulated categories and derived categories* by Jinghui Yang & Shuwei Wang. **Warning.** Currently these notes grew out from a talk and was not self-contained in nature. In the future they may be extended to a more inclusive version, where I aim to present derived categories and localisations rigourously.

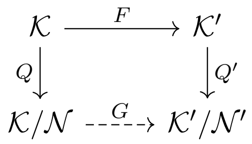
0 Derived Functors

This section mainly follows [李文威]. The relevant sections are 1.8, 1.11, 3.2, 4.6–4.9, 4.12.

Recall that from an Abelian category $\mathcal{A}$ we can build the **homotopy category** $\mathsf{K}(\mathcal{A})$ by taking quotient by chain maps homotopic to zero in the chain complex category $\mathsf{Ch}(\mathcal{A})$, and the **derived category** $\mathsf{D}(\mathcal{A})$ by (Verdier) localisation on the acyclic complexes in $\mathsf{K}(\mathcal{A})$. In particular, every quasi-isomorphism of chains in $\mathcal{A}$ becomes an isomorphism in $\mathsf{D}(\mathcal{A})$ (and $\mathsf{D}(\mathcal{A})$ is universal with respect to this property by construction). In general, $\mathsf{K}(\mathcal{A})$ and $\mathsf{D}(\mathcal{A})$ are not Abelian, but rather **triangulated categories**. For all the technical details we refer to the notes from the previous talk. If $\mathcal{A}$ has enough injectives, then $\mathsf{D}^+(\mathcal{A})$ is equivalent to $\mathcal{I}_{\mathcal{A}}$, the full subcategory of injective objects of $\mathcal{A}$.

There is a natural way to define derived functor under the viewpoint of derived categories. First we recall the classical definition. Suppose that $\mathcal{A}$ is an Abelian category with enough injectives. For $A \in \mathrm{Obj}(\mathcal{A})$, let $A \to I^\bullet$ be an injective resolution of A. Suppose that $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor. Then the ***n*-th right derived functor** of F acting on X is given by $\mathsf{R}^n F(A) := \mathrm{H}^n(F(I^\bullet))$.

Let $\mathcal{K}$ and $\mathcal{K}'$ be triangulated categories, and $Q\colon \mathcal{K} \to \mathcal{K}/\mathcal{N}$ and $Q'\colon \mathcal{K}' \to \mathcal{K}'/\mathcal{N}'$ be Verdier localisations. Suppose that $F\colon \mathcal{K} \to \mathcal{K}'$ is a triangulated functor (i.e. preserving distinguished triangles). The naive idea is to seek for a functor G such that the following diagram commutes (and satisfies some universal properties):

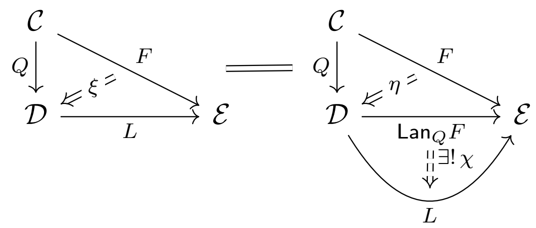


For this we need the Kan extension from category theory. Let's recap.

Definition 0.1. Consider functors $Q\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{E}$. The **left Kan extension** of F by Q consists of the following data:

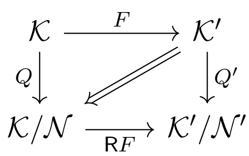
- A functor $\mathsf{Lan}_Q F\colon \mathcal{D} \to \mathcal{E}$;
- A natural transformation $\eta\colon F \Rightarrow \mathsf{Lan}_Q F \circ Q$;

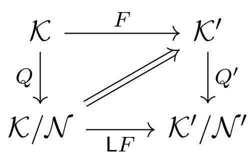
which satisfy the following universal property: for any functor $L\colon \mathcal{D} \to \mathcal{E}$ and natural transformation $\xi\colon F \Rightarrow L \circ Q$, there exists a unique $\chi\colon \mathsf{Lan}_Q F \Rightarrow L$ such that $\xi = (\chi \circ Q) \circ \eta$.



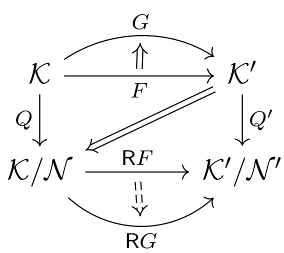
Considering left Kan extension in the opposite categories, we could define **right Kan extension**. The corresponding diagram is given by reversing all natural transformations in the above diagram.

Definition 0.2. Let $F\colon \mathcal{K} \to \mathcal{K}'$ as above. If the left (*resp.* right) Kan extension $\mathsf{Lan}_Q(Q' \circ F)$ (*resp.* $\mathsf{Ran}_Q(Q' \circ F)$) exists and is a triangulated functor, then it is called the right (*resp.* left) **derived functor** of F, denoted by $\mathsf{R}F$ (*resp.* $\mathsf{L}F$).





Remark. Suppose that $G\colon \mathcal{K} \to \mathcal{K}'$ is another triangulated functor with a natural transformation $\eta\colon F \Rightarrow G$. If the right derived functor $\mathsf{R}G$ exists, then there is a canonical natural transformation $\mathsf{R}F \Rightarrow \mathsf{R}G$ by the universal property of right Kan extension.



Then we focus on the derived categories. Note that an additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ between Abelian categories induces the homotopy functor $\mathsf{K}F\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}')$[1] which is triangulated. Consider the Kan extensions:

[1] The cases for K^+, K^-, and K^{b} are identical.

Assuming existence, $\mathsf{R}F$ (*resp.* $\mathsf{L}F$) is called the right (*resp.* left) derived functor of F. Their uniqueness is ensured by the universal property. What about existence?

Definition 0.3. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be as above. Let $\mathcal{J}$ be a triangulated subcategory of $\mathsf{K}(\mathcal{A})$. We say that $\mathcal{J}$ is ***F*-injective** (*resp.* ***F*-projective**), if:

- Resolution: For $X \in \mathrm{Obj}(\mathsf{Ch}(\mathcal{A}))$ there exists $Y \in \mathrm{Obj}(\mathcal{J})$ and a quasi-isomorphism $X \to Y$ (*resp.* $Y \to X$).
- Preserving null system: $F(\mathrm{Obj}(\mathcal{N}(\mathcal{A}) \cap \mathcal{J})) \subseteq \mathrm{Obj}(\mathcal{N}(\mathcal{A}'))$

Note that here the null system $\mathcal{N}(\mathcal{A})$ is the acyclic complexes in $\mathsf{Ch}(\mathcal{A})$.

Remark. There is a similar notion for subcategories of $\mathcal{A}$. Let $\mathcal{I}$ be an additive full subcategory of $\mathcal{A}$. We say that $\mathcal{I}$ is of **type I** (*resp.* **type P**) relative to F, if:

- For any $X \in \mathrm{Obj}(\mathcal{A})$ there exists $Y \in \mathrm{Obj}(\mathcal{I})$ and a monomorphism $X \to Y$ (*resp.* epimorphism $Y \to X$);
- For any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, if $X, Y \in \mathrm{Obj}(\mathcal{I})$ then $Z \in \mathrm{Obj}(\mathcal{I})$. (*resp.* If $Y, Z \in \mathrm{Obj}(\mathcal{I})$ then $X \in \mathrm{Obj}(\mathcal{I})$.) In this case $0 \to F(X) \to F(Y) \to F(Z) \to 0$ is also exact.

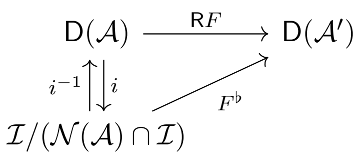
This should be considered as the generalisation of injective objects in $\mathcal{A}$. Indeed the subcategory $\mathcal{I}_{\mathcal{A}}$ of injective objects of $\mathcal{A}$ is of type I relative to any additive functor F.

The terminology is taken from [李文威, 4.8.2]. In fact, this notion is what [Scha, 4.7.5] calls *F-injective*. The two definitions are closely related. If $\mathcal{I} \subseteq \mathcal{A}$ is of type I relative to F, then $\mathsf{K}(\mathcal{I}) \subseteq \mathsf{K}(\mathcal{A})$ is F-injective.

Proposition 0.4

Let $F\colon \mathcal{A} \to \mathcal{A}'$ be as above. Suppose that $\mathsf{K}(\mathcal{A})$ has an F-injective (*resp.* F-projective) subcategory. Then the right (*resp.* left) derived functor $\mathsf{R}F$ (*resp.* $\mathsf{L}F$) exists.

Proof. Let $\mathcal{I}$ be an F-injective subcategory of $\mathsf{K}(\mathcal{A})$. By Theorem 3.5 in [YS], there is an equivalence of category $\mathsf{D}(\mathcal{A}) \simeq \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I})$. Since $F(\mathrm{Obj}(\mathcal{N}(\mathcal{A}) \cap \mathcal{I})) \subseteq \mathrm{Obj}(\mathcal{N}(\mathcal{A}'))$, by the universal property of Verdier localisation there is a functor $F^\flat\colon \mathcal{I}/(\mathcal{N}(\mathcal{A}) \cap \mathcal{I}) \to \mathsf{D}(\mathcal{A}')$. Take $\mathsf{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{A}')$ to be the functor such that the following diagram commutes:



Next we need to verify that $\mathsf{R}F$ is indeed the Kan extension. See [李文威, Prop 1.11.2, Prop 4.6.4]. □

Corollary 0.5

Suppose that $\mathcal{A}$ has enough injectives (*resp.* projectives). Then the right (*resp.* left) derived functor ${}^{+}\mathsf{R}F$ (*resp.* ${}^{+}\mathsf{L}F$) exists for any additive functor $F\colon \mathcal{A} \to \mathcal{A}'$.

Proof. Immediate by [YS, Prop 3.10]. □

Proposition 0.6

Suppose that $\mathcal{A}$ has enough injectives. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be a left exact additive functor. Then for $A \in \mathrm{Obj}(\mathcal{A})$. we have

$$\mathsf{R}^n F(A) = \mathrm{H}^n \circ \mathsf{R}F(QA),$$

where $QA \in \mathsf{D}^+(\mathcal{A})$ and $\mathrm{H}^n\colon \mathsf{D}^+(\mathcal{A}') \to \mathsf{Ab}$ is the n-th cohomology functor.

Proof. Take an injective resolution $A \to I^\bullet$. This gives rise to a quasi-isomorphism $A \to I$ in $\mathsf{K}^+(\mathcal{A})$, where I lies in the F-injective subcategory $\mathsf{K}^+(\mathcal{I}_{\mathcal{A}})$ of $\mathsf{K}^+(\mathcal{A})$. Now we have the isomorphisms

$$\mathsf{R}F(QA) \cong \mathsf{R}F(QI) \cong Q'\mathsf{K}^+F(I).$$

Applying H^n gives the result. □

Proposition 0.7. Long Exact Sequence

Suppose that $F\colon \mathcal{A} \to \mathcal{A}'$ has a right derived functor $\mathsf{R}F$. For any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathsf{D}(\mathcal{A})$, there is a canonical long exact sequence:

$$\cdots \longrightarrow \mathsf{R}^{n-1}(Z) \longrightarrow \mathsf{R}^n F(X) \longrightarrow \mathsf{R}^n F(Y) \longrightarrow \mathsf{R}^n F(Z) \longrightarrow \mathsf{R}^{n+1} F(X) \longrightarrow \cdots$$

Proof. Since $\mathsf{R}F$ is a triangulated functor, the result follows from applying the cohomology functor H^0. □

Comparing to the classical definition, a great advantage of derived functors in this viewpoint is that they compose in a canonical way.

Proposition 0.8

Consider the additive functors among Abelian categories:

$$\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}''$$

Suppose that the right derived functors $\mathsf{R}F$, $\mathsf{R}F'$ and $\mathsf{R}(F' \circ F)$ all exist. Then there is a natural transformation $\mathsf{R}(F' \circ F) \Rightarrow (\mathsf{R}F') \circ (\mathsf{R}F)$.

Moreover, if $\mathcal{I}$ is an F-injective subcategory of $\mathsf{K}(\mathcal{A})$ and $\mathcal{I}'$ is an F'-injective subcategory of $\mathsf{K}(\mathcal{A}')$ such that $F(\mathrm{Obj}(\mathcal{I})) \subseteq \mathrm{Obj}(\mathcal{I}')$, then $\mathcal{I}$ is $F' \circ F$-injective. And the natural transformation above is an isomorphism:

$$\mathsf{R}(F' \circ F) \cong (\mathsf{R}F') \circ (\mathsf{R}F).$$

Proof. For the first part, the natural transformation $\mathsf{R}(F' \circ F) \Rightarrow (\mathsf{R}F') \circ (\mathsf{R}F)$ is induced by the universal property of left Kan extensions (*check it!*) For the second part, take $I \in \mathrm{Obj}(\mathcal{I})$. Using the construction in Proposition 0.4 we obtain

$$(\mathsf{R}F') \circ (\mathsf{R}F)(QI) = Q'' \circ F' \circ F(I) = \mathsf{R}(F' \circ F)(QI)$$

For $X \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}))$, by choosing quasi-isomorphism $X \to I$ we obtain the isomorphism $(\mathsf{R}F') \circ (\mathsf{R}F)(QX) \cong \mathsf{R}(F' \circ F)(QX)$. Finally check that this is compatible with the natural transformation given above. □

Derived Bi-Functors

The tensor functor $- \otimes -$ and the Hom functor $\mathrm{Hom}(-,-)$ are two typical examples of bi-functors of Abelian categories. Since we are interested in these functors, it is useful to treat the derived bi-functors separately.

Definition 0.9. Let $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ be triangulated categories. A bi-functor $F\colon \mathcal{K}_1 \times \mathcal{K}_2 \to \mathcal{K}$ is triangulated, if

- F is triangulated in both slots;
- For any $A \in \mathcal{K}_1$ and $B \in \mathcal{K}_2$, the following diagram anti-commutes[2]:

$$\begin{array}{ccc} F(\mathsf{T}_1 A, \mathsf{T}_2 B) & \longrightarrow & \mathsf{T}F(A, \mathsf{T}_2 B) \\ \downarrow & & \downarrow \\ \mathsf{T}F(\mathsf{T}_1 A, B) & \longrightarrow & \mathsf{T}^2 F(A, B) \end{array}$$

The definition of the left/right derived functor of a triangulated bi-functor is essentially identical. We are interested in the cases where the triangulated categories are homotopy categories of Abelian categories.

Now we consider Abelian categories $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$, where $\mathcal{A}$ admits countable products and coproducts. Let $F\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be an additive bi-functor. Let

$$\mathsf{Ch}_{\oplus} F := \mathrm{Tot}_{\oplus} \circ \mathsf{Ch}^2(F)\colon \mathsf{Ch}(\mathcal{A}_1) \times \mathsf{Ch}(\mathcal{A}_2) \to \mathsf{Ch}(\mathcal{A});$$
$$\mathsf{Ch}_{\Pi} F := \mathrm{Tot}_{\Pi} \circ \mathsf{Ch}^2(F)\colon \mathsf{Ch}(\mathcal{A}_1) \times \mathsf{Ch}(\mathcal{A}_2) \to \mathsf{Ch}(\mathcal{A}).$$

Then induce the triangulated bi-functors $\mathsf{K}_{\oplus} F, \mathsf{K}_{\Pi} F\colon \mathsf{K}(\mathcal{A}_1) \times \mathsf{K}(\mathcal{A}_2) \to \mathsf{K}(\mathcal{A})$.

Let $\mathcal{I}_1, \mathcal{I}_2$ be triangulated subcategories of $\mathsf{K}(\mathcal{A}_1), \mathsf{K}(\mathcal{A}_2)$ respectively. We say that $(\mathcal{I}_1, \mathcal{I}_2)$ is F-injective (*resp.* F-projective), if $\mathcal{I}_2$ is $F(A_1, -)$-injective for any $A_1 \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}_1))$, and $\mathcal{I}_1$ is $F(-, A_2)$-injective for any $A_2 \in \mathrm{Obj}(\mathsf{K}(\mathcal{A}_2))$.

[2] The term is used in [李文威]. It means that the two composite morphisms in the square differ by a sign.

Proposition 0.10

Let $F\colon \mathcal{A}_1 \times \mathcal{A}_2 \to \mathcal{A}$ be as above.

1. If $(\mathcal{I}_1, \mathcal{I}_2)$ is F-injective, then $\mathsf{R}F := \mathsf{RK}_{\Pi}F$ exists. We call it the right derived functor of F;
2. If $(\mathcal{P}_1, \mathcal{P}_2)$ is F-projective, then $\mathsf{L}F := \mathsf{LK}_{\oplus}F$ exists. We call it the left derived functor of F.

Ext and $\mathsf{R}\operatorname{Hom}$

Recall that in *C2.2 Homological Algebra.* we define the $\operatorname{Ext}^n_{\mathcal{A}}(A,B)$ to be the n-th right derived functor of $\operatorname{Hom}_{\mathcal{A}}(A,-)$ acting on $B \in \operatorname{Obj}(\mathcal{A})$. If $\mathcal{A}$ has enough injectives or projectives, then $\operatorname{Ext}^n_{\mathcal{A}}(A,B)$ is computed by an injective resolution $B \to I^\bullet$ of B or a projective resolution $P^\bullet \to A$ of A. By acyclic assembly lemma, $\operatorname{Ext}^n_{\mathcal{A}}(A,B)$ can also be computed as the n-th cohomology of the total complex $\operatorname{Tot}^{\Pi}(\operatorname{Hom}_{\mathcal{A}}(P_\bullet, Q_\bullet))$ using projective resolutions $P_\bullet \to A$ and $Q_\bullet \to B$.

Using the derived category, the Ext group can be defined without using injective or projective resolutions:

Definition 0.11. Let $\mathcal{A}$ be an Abelian category. For chain complexes A, B in $\mathsf{Ch}(\mathcal{A})$, we define the **(hyper-)Ext** group as

$$\operatorname{Ext}^n_{\mathcal{A}}(A,B) := \operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]).$$

This definition gives an obvious multiplication structure on Ext:

$$\begin{aligned} \operatorname{Ext}^n_{\mathcal{A}}(B,C) \times \operatorname{Ext}^m_{\mathcal{A}}(A,B) &\longrightarrow \operatorname{Ext}^{n+m}_{\mathcal{A}}(A,C) \\ (f,g) &\longmapsto f[m] \circ g \end{aligned}$$

In particular it makes $\operatorname{Ext}^\bullet_{\mathcal{A}}(A,A)$ a graded ring for any $A \in \operatorname{Obj}(\mathcal{A})$.

Next we will consider Ext as the right derived functor of Hom bi-functor $\operatorname{Hom}_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Ab}$. It induces the functor on the double complexes:

$$\operatorname{Hom}^{\bullet,\bullet}_{\mathcal{A}}(-,-)\colon \mathsf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Ab}) \times \mathsf{Ch}(\mathsf{Ab}).$$

Define $\mathsf{Ch}\operatorname{Hom}_{\mathcal{A}}(-,-) := \operatorname{Tot}_{\Pi}\operatorname{Hom}^{\bullet,\bullet}_{\mathcal{A}}(-,-)\colon \mathsf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathsf{Ab})$. It is not hard to verify that $\mathsf{Ch}\operatorname{Hom}_{\mathcal{A}}$ is naturally isomorphic to the **Hom complex** $\operatorname{Hom}^\bullet_{\mathcal{A}}$:

$$\operatorname{Hom}^n_{\mathcal{A}}(A,B) := \prod_{k\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(A^k, B^{k+n}), \qquad \mathrm{d}^n_{\mathrm{Hom}}(f) := \mathrm{d}_B \circ f - (-1)^n f \circ \mathrm{d}_A.$$

Lemma 0.12

$$\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(A, B[n]) \cong \mathrm{H}^n(\operatorname{Hom}^\bullet_{\mathcal{A}}(A,B), \mathrm{d}^\bullet_{\mathrm{Hom}}).$$

Proof. Trivial by definition. □

The bi-functor $\mathsf{Ch}\operatorname{Hom}_{\mathcal{A}}$ or $\operatorname{Hom}^{\bullet}_{\mathcal{A}}$ induces the triangulated bi-functor

$$\mathsf{K}\operatorname{Hom}_{\mathcal{A}}\colon \mathsf{K}^{-}(\mathcal{A})^{\mathrm{op}} \times \mathsf{K}^{+}(\mathcal{A}) \to \mathsf{K}^{+}(\mathsf{Ab}).$$

If $\mathcal{A}$ has enough injectives or projectives, then the right derived functor

$$\mathsf{R}\operatorname{Hom}_{\mathcal{A}}\colon \mathsf{D}^{-}(\mathcal{A})^{\mathrm{op}} \times \mathsf{D}^{+}(\mathcal{A}) \to \mathsf{D}^{+}(\mathsf{Ab})$$

exists.

Proposition 0.13

Suppose that $\mathcal{A}$ has enough injectives or projectives. For $A \in \operatorname{Obj}(\mathsf{D}^{-}(\mathcal{A}))$ and $B \in \operatorname{Obj}(\mathsf{D}^{+}(\mathcal{A}))$, there exists a canonical isomorphism

$$\mathrm{H}^{n}\,\mathsf{R}\operatorname{Hom}_{\mathcal{A}}(A, B) \cong \operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]).$$

Proof. Taking the right derived functor in the previous lemma and note that the cohomology functor H^{n} factors through the derived functor. □

Corollary 0.14

Suppose that $\mathcal{A}$ has enough injectives. Let $A, B \in \operatorname{Obj}(\mathcal{A})$ (viewed as complexes concentrated at degree 0). Then there is a canonical isomorphism

$$\operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(A, B[n]) \cong \mathsf{R}^{n}\operatorname{Hom}(A, -)(B)$$

Therefore the hyper-Ext is a generalisation of the usual Ext.

Tor and $\otimes^{\mathsf{L}}$

In this part we only consider R-modules. For $A, B \in \mathsf{Ch}(R\text{-}\mathsf{Mod})$, from *C3.1 Algebraic Topology* we recall the tensor product of complexes is given by the total complex $A \otimes_R B := \operatorname{Tot}_{\oplus}(A^{\bullet} \otimes_R B^{\bullet})$.

Definition 0.15. For $A, B \in \mathsf{Ch}(R\text{-}\mathsf{Mod})$, the **total tensor product** of A and B is the left derived functor

$$A \otimes^{\mathsf{L}}_{R} B := \mathsf{L}(- \otimes_R -)(A, B).$$

$\mathsf{L}(- \otimes_R -)\colon \mathsf{D}^{-}(\mathsf{Mod}\text{-}R) \times \mathsf{D}^{-}(R\text{-}\mathsf{Mod}) \to \mathsf{D}^{-}(\mathsf{Ab})$ exists because R-Mod has enough projectives. By taking cohomology we have the **(hyper-)Tor** groups:

$$\operatorname{Tor}^{R}_{n}(A, B) := \mathrm{H}_{n}(A \otimes^{\mathsf{L}}_{R} B).^{3}$$

Similar as hyper-Ext, using the theory of derived functors we can verify that the hyper-Tor reduces to the usual Tor on $\operatorname{Obj}(R\text{-}\mathsf{Mod})$ (defined using projective resolutions).

Remark. In general $\mathsf{QCoh}(X)$ does not have enough projectives. We will have to instead use flat resolutions to compute the total tensor product. See later.

[3] Cohomology and homology make no difference in algebra. By convention, $\mathrm{H}_{n} := \mathrm{H}^{-n}$.

Proposition 0.16. Derived Tensor-Hom Adjunction

Let $A \in \mathsf{D}(\mathsf{Mod}\text{-}R)$, $B \in \mathsf{D}(R\text{-}\mathsf{Mod})$, and $C \in \mathsf{D}(\mathsf{Ab})$. There are canonical isomorphisms in $\mathsf{D}(\mathsf{Ab})$:

$$\begin{aligned}\mathsf{R}\operatorname{Hom}_{\mathsf{Ab}}(X \otimes^{\mathsf{L}}_R Y, Z) &\cong \mathsf{R}\operatorname{Hom}_{\mathsf{Mod}\text{-}R}(X, \mathsf{R}\operatorname{Hom}_{\mathsf{Ab}}(Y, Z)) \\ &\cong \mathsf{R}\operatorname{Hom}_{R\text{-}\mathsf{Mod}}(Y, \mathsf{R}\operatorname{Hom}_{\mathsf{Ab}}(X, Z)).\end{aligned}$$

1 Sheaves of Modules

Let us recall some basic algebraic geometry from *C2.6 Introduction to Schemes*. All rings are commutative with multiplicative identity 1.

Definition 1.1. A **scheme** $(X, \mathcal{O}_X)$ is a locally ringed space such that for any $x \in X$ there exists an open neighbourhood $U \in \mathsf{Top}(X)$ of x such that $(U, \mathcal{O}_X\big|_U) \cong (\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R.

Example 1.2. A **variety** over a field k is a reduced[4], separated[5], finite type[6] scheme over k. An **affine variety** is a closed subscheme of $\mathbb{A}^n := \operatorname{Spec} k[x_1, ..., x_n]$. A **projective variety** is a reduced closed subscheme of $\mathbb{P}^n := \operatorname{Proj} k[x_0, ..., x_n]$. A **quasi-projective variety** is an open subscheme of a projective variety.

Definition 1.3. Let $(X, \mathcal{O}_X)$ be a scheme. A **sheaf of $\mathcal{O}_X$-modules** F on X is a sheaf $F\colon \mathsf{Top}(X)^{\mathrm{op}} \to \mathsf{Ab}$ such that:

- For any $U \in \mathsf{Top}(X)$, $F(U)$ is a $\mathcal{O}_U$-module;
- The module structure is compatible with restrictions on X.

The category of $\mathcal{O}_X$-modules is denoted by $\mathcal{O}_X\text{-}\mathsf{Mod}$. It is an Abelian category with enough injectives.

Recall the way we construct the affine scheme $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ from any ring R. For any R-module M, we can construct the sheaf $\widetilde{M} \in \operatorname{Obj}(\mathcal{O}_{\operatorname{Spec} R}\text{-}\mathsf{Mod})$ in a similar way (see the course notes for details). In particular we have the stalks $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} R$ and the global sections $\widetilde{M}(\operatorname{Spec} R) = M$. For a general scheme X, $\widetilde{M}$ can be constructed from an $\mathcal{O}_X(X)$-module M.

Definition 1.4. Let $F \in \mathcal{O}_X\text{-}\mathsf{Mod}$. We say that F is **quasi-coherent**, if it satisfies any of the following equivalent conditions:

1. F is **locally presented**. That is, for any $x \in X$ there exists a neighbourhood $U \in \mathsf{Top}(X)$ of x such that there exists an exact sequence of the following form:
$$\bigoplus_{i \in I} \mathcal{O}_U \longrightarrow \bigoplus_{j \in J} \mathcal{O}_U \longrightarrow F\big|_U \longrightarrow 0$$
2. For any $x \in X$ there exists an affine neighbourhood $U \cong \operatorname{Spec} R \ni x$ such that $F\big|_U \cong \widetilde{M}$ for some R-module M.
3. There exists an affine open cover $\{U_i\}_{i \in I}$ of X such that $F\big|_{U_i} \cong \widetilde{M_i}$ for R_i-modules M_i, where $\operatorname{Spec} R_i \cong U_i$.

[4] i.e. all rings $\mathcal{O}_X(U)$ are reduced rings.
[5] i.e. the diagonal morphism $\Delta\colon X \to X \times_{\operatorname{Spec} k} X$ is a closed immersion.
[6] i.e. quasi-compact and all open affine rings are finite type over k.

If additionally for each U_i in (3), $F(U_i)$ is a finitely generated $\mathcal{O}_{U_i}$-module, then we say that F is **coherent**. The category of quasi-coherent (*resp.* coherent) sheaves is denoted by $\mathsf{QCoh}(X)$ (*resp.* $\mathsf{Coh}(X)$).

Definition 1.5. Let $F \in \mathcal{O}_X\text{-}\mathsf{Mod}$. We say that F is a **vector bundle** (i.e. locally free of finite rank) if for $x \in X$ there exists an open neighbourhood $U \in \mathsf{Top}(X)$ of x such that $F\big|_U \cong \mathcal{O}_U^{\oplus n}$. The category of vector bundles is denoted by $\mathsf{Vect}(X)$. F is called a **line bundle** (or invertible sheaf) if additionally $n = 1$ for all $x \in X$.

Remark. For a coherent sheaf F on X, if the stalk takes the form $F_x \cong \mathcal{O}_{X,x}^{\oplus n(x)}$ for any $x \in X$, then F is a vector bundle. In particular, $\mathsf{Vect}(X)$ is a full subcategory of $\mathsf{Coh}(X)$ if X is locally Noetherian (i.e. every open affine ring is Noetherian).

Why do we want quasi-coherence?

- $\mathsf{Coh}(X)$ and $\mathsf{QCoh}(X)$ are Abelian categories, but $\mathsf{Vect}(X)$ is not Abelian in general.
- When $X = \operatorname{Spec} R$, $M \mapsto \widetilde{M}$ gives an equivalence of categories $R\text{-}\mathsf{Mod} \simeq \mathsf{QCoh}(X)$.
- Pull-backs preserve quasi-coherence. If X is Noetherian, then push-forwards also preserve quasi-coherence.
- If X is Noetherian, then $\mathsf{QCoh}(X)$ has enough injectives. (*Let's prove it below!*)
- If X and Y are projective varieties, then $\mathsf{Coh}(X) \simeq \mathsf{Coh}(Y)$ implies $X \cong Y$.

Slogan. Quasi-coherent (*resp.* coherent) sheaves are the analogue of modules (*resp.* finitely generated modules) over a ring.

Functors of Sheaves of Modules

There are some constructions in $\mathcal{O}_X\text{-}\mathsf{Mod}$.

- **Coproduct**: $\bigoplus_{i \in J} F_i$ is the sheafification of the presheaf $U \mapsto \bigoplus_{i \in J} F_i(U)$;
- **Tensor product**: $F \otimes_{\mathcal{O}_X} G$ is the sheafification of the presheaf $U \mapsto F(U) \otimes_{\mathcal{O}_U} G(U)$.
- **Hom sheaf**: $\mathcal{H}om_{\mathcal{O}_X}(F, G)$ is the presheaf $U \mapsto \operatorname{Hom}_{\mathcal{O}_U}(F\big|_U, G\big|_U)$, which is already a sheaf.
- **Dual sheaf**: $F^\vee := \mathcal{H}om_{\mathcal{O}_X}(F, \mathcal{O}_X)$.

Definition 1.6. Let $f\colon X \to Y$ be a morphism of schemes. Let $F \in \operatorname{Obj}(\mathcal{O}_X\text{-}\mathsf{Mod})$ and $G \in \operatorname{Obj}(\mathcal{O}_Y\text{-}\mathsf{Mod})$.

1. The **direct image** (or push-forward) f_*F of F is a $\mathcal{O}_Y$-module given by $U \mapsto F(f^{-1}(U))$;
2. The **pull-back** f^*G of G is a $\mathcal{O}_X$-module given by $f^*G = f^{-1}(G) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.

The key observation is the adjunction $f^* \dashv f_*$: there is a canonical isomorphism

$$\operatorname{Hom}_{\mathcal{O}_X}(f^*G, F) \cong \operatorname{Hom}_{\mathcal{O}_Y}(G, f_*F).$$

So it is natural to talk about the derived functors of f_* and f^*.

Now let us derive some functors!

Functors	Derived functors	n-th derived functors
Global sections $\Gamma(X,-)\colon \mathsf{Ab}(X) \to \mathsf{Ab}$	$\mathsf{R}\Gamma(X,-)$	Sheaf cohomology $\mathrm{H}^n(X,-)$
$\mathrm{Hom}_{\mathcal{O}_X}(-,-)\colon (\mathcal{O}_X\text{-}\mathsf{Mod})^{\mathrm{op}} \times \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathsf{Ab}$	$\mathsf{R}\,\mathrm{Hom}_{\mathcal{O}_X}(-,-)$	Ext group $\mathrm{Ext}^n_X(-,-)$
$\mathcal{H}om_{\mathcal{O}_X}(-,-)\colon (\mathcal{O}_X\text{-}\mathsf{Mod})^{\mathrm{op}} \times \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathcal{O}_X\text{-}\mathsf{Mod}$	$\mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(-,-)$	Ext sheaf $\mathcal{E}xt^n_X(-,-)$
$-\otimes_{\mathcal{O}_X} -\colon \mathcal{O}_X\text{-}\mathsf{Mod} \times \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathcal{O}_X\text{-}\mathsf{Mod}$	$-\otimes^{\mathsf{L}}_{\mathcal{O}_X} -$	Tor group $\mathrm{Tor}^X_n(-,-)$
$f_*\colon \mathcal{O}_X\text{-}\mathsf{Mod} \to \mathcal{O}_Y\text{-}\mathsf{Mod}$	$\mathsf{R}f_*$	Higher direct image $\mathsf{R}^n f_*$
$f^*\colon \mathcal{O}_Y\text{-}\mathsf{Mod} \to \mathcal{O}_X\text{-}\mathsf{Mod}$	$\mathsf{L}f^*$	$\mathsf{L}_n f^*$

Derived Categories of Coherent Sheaves

We will always assume that X is Noetherian[7]. A good new and a bad news.

Proposition 1.7

Let X be a Noetherian scheme. Then $\mathsf{QCoh}(X)$ has enough injectives.

Proof. [HartsAG, Cor III.3.6] Cover X with a finite number of affine opens $U_i = \operatorname{Spec} A_i$, and let $F|_{U_i} = \widetilde{M_i}$ for each i. Embed M_i in an injective A_i-module I_i. For each i, let $f : U_i \to X$ be the inclusion, and let $G = \bigoplus_i f_*(\widetilde{I_i})$. For each i we have an injective map of sheaves $F|_{U_i} \to \widetilde{I_i}$. Hence we obtain a map $F \to f_*(\widetilde{I_i})$. Taking the direct sum over i gives a map $F \to G$ which is clearly injective. Check that G is flasque[8] and quasi-coherent. G is an injective object in $\mathsf{QCoh}(X)$. □

Remark. Alternatively it can also be shown that $\mathsf{QCoh}(X)$ is a **Grothendieck category** (see [李文威, §2.10]), thus having enough injectives.

In general $\mathsf{Coh}(X)$ does not have enough injectives. Think of $X = \operatorname{Spec}\mathbb{Z}$, where $\mathsf{Coh}(X)$ is the category of finitely generated Abelian groups. Instead of $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X)$, we instead work with the full subcategory $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ of $\mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)$:

$$\mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)) := \left\{F \in \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)\colon\ \mathrm{H}^n(F) \in \mathrm{Obj}(\mathsf{Coh}(X));\ \mathrm{H}^i(F) = 0 \text{ for } |i| \gg 0\right\}.$$

In general for a full Abelian subcategory $\mathcal{A} \subseteq \mathcal{B}$, the derived categories $\mathsf{D}(\mathcal{A})$ and $\mathsf{D}_{\mathcal{A}}(\mathcal{B})$ could be quite different. However we have the following

Proposition 1.8

Let X be a Noetherian scheme. The natural functor $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \to \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)$ defines a triangulated equivalence of categories

$$\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \simeq \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X).$$

Proof. [Huyb, Prop 3.5] It is clear that $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \to \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X)$ is fully faithful. It suffices to show essential surjectivity. Consider a bounded complex of quasi-coherent sheaves with coherent cohomology:

[7] i.e. quasi-compact and every open affine ring is Noetherian.
[8] i.e. restriction maps of F are surjective.

$$0 \longrightarrow F^n \longrightarrow \cdots \longrightarrow F^m \longrightarrow 0$$

By induction suppose F^j is coherent for $j > i$. Consider the surjections $\mathrm{d}^i \colon F^i \to \operatorname{im} \mathrm{d}^i \subseteq F^{i+1}$ and $\ker \mathrm{d}^i \to \mathrm{H}^i(F^\bullet)$. We can find coherent subsheaves of $F_1^i \subseteq F^i$ and $F_2^i \subseteq \ker \mathrm{d}^i \subseteq F^i$ such that the restrictions of the above morphisms are still surjective ([HartsAG, Ex II.5.15]). Now replace F^i by its subsheaf generated by F_1^i and F_2^i, and let F^{i-1} be the preimage under d^{i-1} of the new F^i. Clearly the inclusions induce a quasi-isomorphism of the new complex with the old one and now F^i is also coherent. □

So we can resolve a coherent sheaf by quasi-coherent sheaves injective in $\mathsf{QCoh}(X)$ in order to compute $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X)$.

Derived Functors of Coherent Sheaves

In this part we address some technical issues in passing the functors from $\mathcal{O}_X$-Mod to $\mathsf{Coh}(X)$. We follow [Huyb §3.3]. A lot of relevant results are scattered in Chapter III of [HartsAG]...

Theorem 1.9. Grothendieck Vanishing Theorem

Let X be a Noetherian topological space of dimension n. Then $\mathrm{H}^i(X, F) = 0$ for all $F \in \operatorname{Obj}(\mathsf{Ab}(X))$ and $i > n$.

Proof. See [HartsAG Thm III.2.7]. □

Theorem 1.10

Let F be a coherent sheaf on a scheme X which is proper (e.g. projective) over a field k. Then $\mathrm{H}^i(X, F)$ is finite dimensional over k for all i.

Proof. See [HartsAG Thm III.5.2]. □

Corollary 1.11

Let X be a projective variety over a field k. The global section functor $\Gamma(X, -)$ is a left exact functor $\mathsf{Coh}(X) \to k\text{-}\mathsf{Mod}^{\mathrm{fd}}$. The right derived functor $\mathsf{R}\Gamma$ can be computed via the composition $\mathsf{D}^{\mathrm{b}}\mathsf{Coh}(X) \simeq \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \hookrightarrow \mathsf{D}^{\mathrm{b}}\mathsf{QCoh}(X) \to \mathsf{D}^{\mathrm{b}}(k\text{-}\mathsf{Mod})$.

Theorem 1.12

1. Let $f \colon X \to Y$ be a morphism of Noetherian schemes. Let F be a quasi-coherent sheaf over X. The higher direct images $\mathsf{R}^i f_*(F) = 0$ for $i > \dim X$.
2. Let $f \colon X \to Y$ be a proper morphism of Noetherian schemes. Let F be a coherent sheaf over X. The higher direct images $\mathsf{R}^i f_*(F)$ are also coherent for all i.

Proof. See [HartsAG Thm III.8.1 III.8.8]. □

Corollary 1.13

Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. The direct image $f_*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(Y)$ induces the right derived functor $\mathsf{R}f_*\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$.

2 Coherent Sheaves on a Smooth Projective Variety

Smoothness and Serre Duality

Let k be an algebraically closed field. Recall that in *C3.4 Algebraic Geometry* we define the non-singular points of a quasi-projective variety by counting the dimension of (co)tangent space at that point:

Definition 2.1. A scheme X is **non-singular** (or regular)[9] at $x \in X$ if $\mathcal{O}_{X,x}$ is a regular local ring. That is, $\dim_{\mathcal{O}_{X,x}/\mathfrak{m}_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$. X is non-singular if it is non-singular at all points[10].

The non-singularity can be characterised by Kähler differentials, which is the algebraic analogue of the cotangent bundle.

Proposition 2.2

Let X be an irreducible variety over k. Then X is regular if and only if the sheaf of Kähler differentials $\Omega_{X/k}$ is a vector bundle over X of dimension $n = \dim X$.

Proof. See [HartsAG Thm II.8.15]. □

Definition 2.3. Let X be a non-singular irreducible variety over k. Let $n = \dim X$. We define the

- **tangent sheaf** $\mathcal{T}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X)$, which is a vector bundle of rank n;
- **canonical sheaf** $\omega_X := \bigwedge^n \Omega_{X/k}$, which is a line bundle.

Perfect Complexes

Definition 2.4. Let $F \in \mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X))$. We say that F is a **strictly perfect complex**, if F is quasi-isomorphic to a bounded complex of vector bundles on X. We say that F is a **perfect complex** if there exists an affine cover $\{U_i\}_{i\in I}$ of X such that each $F\big|_{U_i}$ is quasi-isomorphic to some strictly perfect complex F_i on U_i.

The perfect complexes form a full subcategory $\mathsf{Perf}(X)$ of $\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$.

Proposition 2.5. Smoothness via Perfect Complexes

Suppose that X is a Noetherian scheme. Then X is regular if and only if the inclusion $\mathsf{Perf}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ is an equivalence of categories.

[9] It is bad to use the term *smooth* here, as it is reserved for a property of morphisms.

[10] Equivalently at all closed points, because the stalk at any non-closed point is a localisation of the stalk at a closed point, and localisation preserves regular local rings.

Proof. Idea: On a regular scheme X, any coherent sheaf F admits a locally free resolution of length $\dim X$. This is the generalisation of the affine result: $\operatorname{Spec} R$ is an n-dimensional regular affine variety if and only if every (finitely generated) R-module M admits a (finitely generated) projective resolution of length n. □

Remark. For a general variety X, we may introduce the quotient category (*localisation?*)

$$\mathsf{Sing}(X) := \mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(X)/\,\mathsf{Perf}(X)$$

which measures how singular X is. Of course $\mathsf{Sing}(X)$ is trivial if X is regular.

By passing to $\mathsf{Perf}(X)$ we will be able to define the bounded version of $\mathsf{R}\mathcal{H}om$ and $\otimes^{\mathsf{L}}$ for coherent sheaves when X is a smooth projective variety. In particular, for $F \in \operatorname{Obj}(\mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(X))$, the **derived dual**

$$F^\vee := \mathsf{R}\mathcal{H}om(F, \mathcal{O}_X) \in \mathsf{D}^+\mathsf{QCoh}(X)$$

is in $\mathsf{D}^{\mathsf{b}}_{\mathsf{Coh}}(X)$ when X is regular.

Serre Duality

Theorem 2.6. Serre Duality

Let X be a n-dimensional smooth projective variety over k with canonical sheaf ω_X. For $F \in \operatorname{Obj}(\mathsf{Vect}(X))$, there are functorial isomorphisms of vector spaces

$$\mathrm{H}^i(X, F)^\vee \cong \operatorname{Ext}^{n-i}_X(F, \omega_X) \cong \mathrm{H}^{n-i}(X, F^\vee \otimes_{\mathcal{O}_X} \omega_X).$$

Proof. See [HartsAG §III.7]. The second isomorphism follows from the general facts $\operatorname{Ext}^n_X(E \otimes_{\mathcal{O}_X} F, G) \cong \operatorname{Ext}^n(E, F^\vee \otimes_{\mathcal{O}_X} G)$ (here F needs to be a vector bundle) and $\operatorname{Ext}^n(\mathcal{O}_X, F) \cong \mathrm{H}^n(X, F)$ for $\mathcal{O}_X$-modules E, F, G. □

Remark. If we take $F = \Omega^p := \bigwedge^p \Omega_{X/k}$ and note that $\Omega^{n-p} \cong (\Omega^p)^\vee \otimes_{\mathcal{O}_X} \omega_X$ ([HartsAG Ex II.5.16.(b)]), then Serre duality takes the form

$$\mathrm{H}^q(X, \Omega^p)^\vee \cong \mathrm{H}^{n-q}(X, \Omega^{n-p}),$$

which is known in complex geometry.

Corollary 2.7

Let X be a n-dimensional smooth projective variety over k. Then $\mathsf{Coh}(X)$ has **global homological dimension** n. That is, $\operatorname{Ext}^i_X(F, G) = 0$ for $i > n$ and any coherent sheaves F, G.

Remark. In particular, for a smooth projective curve C, $\mathsf{Coh}(C)$ has global homological dimension 1. It can be proven that every $F \in \mathsf{D}^{\mathsf{b}}\mathsf{Coh}(C)$ is quasi-isomorphic to its cohomology:

$$F \cong \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(F)[-i].$$

Serre Functor

Let us rephrase Serre duality using some category theory.

Definition 2.8. Let $\mathcal{A}$ be a k-linear category. A **Serre functor** $S\colon \mathcal{A} \to \mathcal{A}$ is a k-linear equivalence such that for $A, B \in \mathrm{Obj}(\mathcal{A})$ there exists a functorial isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \cong \mathrm{Hom}_{\mathcal{A}}(B, S(A)).$$

Lemma 2.9

Let $\mathcal{A}$ and $\mathcal{B}$ be k-linear categories with finite-dimensional Hom spaces. Suppose that they admit Serre functors $S_{\mathcal{A}}$ and $S_{\mathcal{B}}$ respectively. Then any k-linear equivalence $F\colon \mathcal{A} \to \mathcal{B}$ commutes with the Serre functors: $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$.

Proof. This is an application of the Yoneda lemma: since F is fully faithful, one has for any two objects $A, B \in \mathcal{A}$

$$\mathrm{Hom}(A, S_{\mathcal{A}}B) \cong \mathrm{Hom}(FA, FS_{\mathcal{A}}B), \qquad \mathrm{Hom}(B, A) \cong \mathrm{Hom}(FB, FA).$$

Together with the two isomorphisms

$$\mathrm{Hom}(A, S_{\mathcal{A}}B) \cong \mathrm{Hom}(B, A)^{\vee}, \qquad \mathrm{Hom}(FB, FA) \cong \mathrm{Hom}(FA, S_{\mathcal{B}}FB)^{\vee},$$

this yields a functorial isomorphism

$$\mathrm{Hom}(FA, FS_{\mathcal{A}}B) \cong \mathrm{Hom}(FA, S_{\mathcal{B}}FB).$$

Using the hypothesis that F is an equivalence and, in particular, that any object in $\mathcal{B}$ is isomorphic to some $F(A)$, one concludes that there exists a functor isomorphism $F \circ S_{\mathcal{A}} \cong S_{\mathcal{B}} \circ F$. □

Remark. If $\mathcal{A}, \mathcal{B}$ are triangulated categories, then the Serre functors are exact and triangulated.

In particular, Serre functors are useful in inverting adjunction pairs:

Corollary 2.10

Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a k-linear functor. Then

$$G \dashv F \implies F \dashv S_{\mathcal{A}} \circ G \circ S_{\mathcal{B}}^{-1}.$$

Proof. For $A \in \mathrm{Obj}(\mathcal{A})$ and $B \in \mathrm{Obj}(\mathcal{B})$,

$$\mathrm{Hom}_{\mathcal{A}}(A, S_{\mathcal{A}}GS_{\mathcal{B}}^{-1}B) \cong \mathrm{Hom}_{\mathcal{A}}(GS_{\mathcal{B}}^{-1}B, A)^{\vee} \cong \mathrm{Hom}_{\mathcal{B}}(S_{\mathcal{B}}^{-1}B, FA)^{\vee} \cong \mathrm{Hom}_{\mathcal{B}}(FA, B) \quad \square$$

Serre functors gain their name from Serre duality. Indeed, let X be a smooth projective variety. We define the the functor

$$S_X\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X), \qquad F \longmapsto F \otimes_{\mathcal{O}_X} \omega_X[\dim X].$$

Proposition 2.11

The functor S_X defined above is a Serre functor.

Proof. Let $n = \dim X$. let E, F be vector bundles over X. By Serre duality we have

$$\mathrm{Ext}^i_X(E, F) \cong \mathrm{H}^i(X, E^\vee \otimes F) \cong \mathrm{H}^{n-i}(X, E \otimes F^\vee \otimes \omega_X)^\vee \cong \mathrm{Ext}^{n-i}_X(F, E \otimes \omega_X)^\vee.$$

Using Corollary 0.14 we obtain

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(E, F[i]) \cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(F[i], E \otimes \omega_X[n])^\vee \cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(F[i], S_X(E))^\vee.$$

Therefore for any $E, F \in \mathrm{Obj}(\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X))$, we have

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(E, F) \cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)}(F, S_X(E))^\vee.$$ □

Grothendieck–Verdier Duality

The target is to generalise Serre duality to a relative version. Let $f\colon X \to Y$ be a morphism of smooth projective varieties. We define the **relative dimension** $\dim f := \dim X - \dim Y$ and the **relative dualising bundle** $\omega_f := \omega_X \otimes_{\mathcal{O}_X} f^*\omega_Y^{-1}$.

It is impossible to find a right adjoint to the direct image functor $f_*\colon \mathsf{Coh}(X) \to \mathsf{Coh}(Y)$, because we have the adjunction $f^* \dashv f_*$ on the Abelian categories $\mathsf{Coh}(X)$ and $\mathsf{Coh}(Y)$. However it is possible after passing to the derived categories. We can construct $\mathsf{L}f^* \dashv \mathsf{R}f_* \dashv f^!$ by Serre functors.

Theorem 2.12. Grothendieck–Verdier Duality

Let $f\colon X \to Y$ be a morphism of smooth projective varieties. Then the right adjoint of $\mathsf{R}f_*\colon \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X) \to \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$ exists and is given by

$$f^!(F) := \mathsf{L}f^*(F) \otimes_{\mathcal{O}_X} \omega_f[\dim f].$$

Proof. By the previous part it suffices to take $f^! := S_X \circ \mathsf{L}f^* \circ S_Y^{-1}$. □

Grothendieck–Verdier duality has a more general form, which is a functorial isomorphism

$$\mathsf{R}f_* \circ \mathsf{R}\mathcal{H}om_{\mathcal{O}_X}(F, \mathsf{L}f^*(E) \otimes_{\mathcal{O}_X} \omega_f[\dim f]) \cong \mathsf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathsf{R}f_*(F), E)$$

for $F \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(X)$ and $E \in \mathsf{D}^{\mathrm{b}}_{\mathsf{Coh}}(Y)$.