Perron's Formula

Math 531

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Before our proof of Perron's formula, we firstly recall the lemma proved in our last class:

Lemma 0.1

Let $c > 0$, and for $T > 0$ and $y > 0$ set

$$I(y,T) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\mathrm{d}s, \quad I_1(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{y^s}{s(s+1)}\mathrm{d}s.$$

1. Given $T > 0$ and $y > 0$, $y \neq 1$, we have

$$\begin{cases} \left|I(y,T) - 1\right| \leqslant \frac{y^c}{\pi T \ln y} & \text{if } y > 1, \\ \left|I(y,T)\right| \leqslant \frac{y^c}{\pi T |\ln y|} & \text{if } 0 < y < 1. \end{cases} \tag{1}$$

2. For all $y > 0$ we have

$$I_1(y) = \begin{cases} 1 - \frac{1}{y} & \text{if } y > 1, \\ 0 & \text{if } 0 < y \leqslant 1. \end{cases} \tag{2}$$

From now on we will use this conclusion without proving it. Now we define some notations used later:

$$M(f,y) = \sum_{n \leqslant y} f(n);$$
$$M_1(f,x) = \int_1^x M(f,y)\mathrm{d}y = \sum_{n \leqslant x} f(n)(x-n).$$

Theorem 0.1 Perron's formula for $M_1(f,x)$

Let $f(n)$ be an arithmetic function, and suppose that the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

has finite abscissa of absolute convergence σ_a, i.e. $\sigma_a < \infty$. Then we have, for any $c > \max(0, \sigma_a)$ and real number $x \geqslant 1$,

$$M_1(f,x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^{s+1}}{s(s+1)}\mathrm{d}s. \tag{3}$$

Note: The integral in (3) is absolutely convergent, since

$$\left|F(s)\right| \leqslant \sum_{n=1}^{\infty} \frac{\left|f(n)\right|}{n^c} < \infty,$$

and on the line of integration we have $\left|x^{s+1}\right| = x^{c+1}$. Thus we conclude that

$$F(s)\frac{x^{s+1}}{s(s+1)} = O\left(\frac{x^{c+1}}{|s|^2}\right),$$

which proves the claim.

Proof: In order to interchange the order of integration and summation, we first note that

$$\begin{aligned}\int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \left|\frac{\left|f(n)\right|}{n^s}\right| \left|\frac{x^{s+1}}{s(s+1)}\right| \cdot |\mathrm{d}s| \\ \leqslant x^{c+1} \sum_{n=1}^{\infty} \frac{\left|f(n)\right|}{n^c} \int_{c-i\infty}^{c+i\infty} \frac{x^{c+1}}{\left|s(s+1)\right|} \cdot |\mathrm{d}s| < \infty,\end{aligned}$$

since, by the assumption $c > \max(0, \sigma_a)$, we have $\sum_{n=1}^{\infty} \frac{|f(n)|}{n^c} < \infty$ and

$$\int_{c-i\infty}^{c+i\infty} \frac{1}{\left|s(s+1)\right|} \cdot |\mathrm{d}s| \leqslant \int_{c-i\infty}^{c+i\infty} \frac{1}{|s|^2} \cdot |\mathrm{d}s| = \int_{-\infty}^{+\infty} \frac{1}{c^2+t^2}\mathrm{d}t < \infty,$$

and so we're done. Now interchanging the integration and summation in (3), we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\frac{x^{s+1}}{s(s+1)}\mathrm{d}s &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \frac{f(n)}{n^s}\frac{x^{s+1}}{s(s+1)}\mathrm{d}s \\ &= \frac{1}{2\pi i}\sum_{n=1}^{\infty} xf(n) \int_{c-i\infty}^{c+i\infty} \frac{(x/n)^s}{s(s+1)}\mathrm{d}s \\ &= \sum_{n=1}^{\infty} xf(n)I_1(x/n) \\ &= \sum_{n\leqslant x} f(n)(x-n) = M_1(f,x).\end{aligned}$$

□

There is another formula related to $M(f,x)$:

Theorem 0.2 Perron's formula for $M(f,x)$

Let $f(n)$ be an arithmetic function, and suppose that the Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{f(n)}{n^s}$$

has finite abscissa of absolute convergence σ_a, i.e. $\sigma_a<\infty$. Then we have, for any $c>\max(0,\sigma_a)$ and any non-integral value $x>1$,

$$M(f,x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)\frac{x^s}{s}\mathrm{d}s, \tag{4}$$

where the improper integral $\int_{c-i\infty}^{c+i\infty}=\lim_{T\to\infty}\int_{c-iT}^{c+iT}$. Moreover, given $T>0$, we have

$$M(f,x)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}F(s)\frac{x^s}{s}\mathrm{d}s+R(T), \tag{5}$$

where

$$|R(T)|\leqslant\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|f(n)|}{n^c\,|\ln(x/n)|}. \tag{6}$$

Proof: It suffices to prove the formula (5). We use the same method in the proof of the former theorem. Since the range $[c-iT,c+iT]$ is compact, the Dirichlet series $F(s)=\sum_{n=1}^{\infty}f(n)n^{-s}$ converges on that interval absolutely and uniformly. Thus we can interchange the order of integration and summation. Namely,

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}F(s)\frac{x^s}{s}\mathrm{d}s&=\sum_{n=1}^{\infty}f(n)I(x/n,T)\\&=\sum_{n\leqslant x}f(n)+E(T),\end{aligned}$$

where

$$|E(T)|\leqslant\sum_{n=1}^{\infty}|f(n)|\frac{(x/n)^c}{T\,|\ln(x/n)|}=\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|f(n)|}{n^c\,|\ln(x/n)|}.$$

Taking $R(T)=-E(T)$ in (5) we conclude our proof. □

Note: The restriction to non-integral values of x in (4) can be dropped if we consider

$$\begin{aligned}I(1,T)=\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\mathrm{d}s}{s}&=\frac{1}{2\pi}\int_{-T}^{T}\frac{c-it}{c^2+t^2}\mathrm{d}t\\&=\frac{1}{2\pi}\int_{-T}^{T}\frac{c}{c^2+t^2}\mathrm{d}t=\frac{1}{2\pi}\int_{-T/c}^{T/c}\frac{1}{1+u^2}\mathrm{d}u\\&=\frac{1}{2\pi}\left[\arctan(T/c)-\arctan(-T/c)\right],\end{aligned}$$

which converges to $1/2$ as $T \to \infty$. So we can take $M^*(f,x) = \frac{1}{2}\left(M(f,x^-) + M(f,x^+)\right)$, where $+, -$ denotes the right and left limit respectively, in replace of $M(f,x)$ to achieve this goal.

However, in applications the stated version is sufficient, since for any integer N, $M(f,N)$ is equal to $M(f,x)$ for $N < x < N+1$ and one can therefore apply the formula with such a non-integral value of x. Usually one takes x to be of the form $x = N + 1/2$ in order to minimize the effect a small denominator $\log(x/n)$ on the right-hand side of (6) can have on the estimate. Usually, we take $x = \lfloor x \rfloor + \frac{1}{2}$.