ALGEBRAIC K-THEORY AND TRACE METHOD

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Abstract. Let A be an associative, commutative and unital ring. We introduce the Hochschild homology of A and give the theory of computing its algebraic K-theory groups $K_n(A)$, $n \geq 0$ via the Dennis trace map $D : K(A) \to HH(A)$. We also establish the topological Hochschild homology (THH) and the topological cyclic homology (TC). Working in the category of orthogonal S^1-spectra, we construct the p-cyclotomic spectra and the cyclotomic spectra. The rest of the paper is dedicated to giving a brief description of the lifted topological trace $K(A) \to TC(A) \to THH(A)$.

Contents

1. Introduction

Algebraic K-theory was first introduced by Grothendieck in order to describe his Riemann-Roch theorem in algebraic geometry. It became a branch of modern algebra and played an important role in number theory, algebraic geometry and topology. While it revealed its power in several fields, it was extremely hard to compute, and it often took papers to talk over some fundamental cases. Let A be a ring and denote $K_n(A)$ to be its n^{th} K-theory group. One well-known result for $K_n(\mathbb{F}_p)$ was attained by Quillen in [4]. In general, even if we consider the basic cases when $A = \mathbb{Z}$ or $A = \mathbb{Z}/p^k$, the computation of $K_n(A)$ still remains open. To handle the issue, Goodwillie developed the trace method in his paper [1]. The approach is to recover the information of K-theory from the Dennis trace map $D : K_q(A) \to HH_q(A)$, where $HH_q(A)$ stands for the q^{th} Hochschild homology of A. The Dennis trace map can factor through the negative cyclic homology $HC_q^-(A)$. This led Goodwillie to showed D is an equivalence after tensoring with $\mathbb{Q}$.

In his 1990 ICM address, Goodwillie conjectured that there should be a topological analogue to the idea of trace method. In response to Goodwillie's conjecture, Bökstedt developed the idea of topological Dennis trace map. Later, Bökstedt-Hsiang-Madsen established the topological cyclic homology and the cyclotomic trace map in their study [2]. They lifted the topological Dennis trace $D' : K(A) \to TC(A) \to THH(A)$, where we abuse the notation A to denote the Eilenberg-MacLane spectrum of the ring A. Afterwards, McCarthy in [3] proved that the cyclotomic trace map would generate an equivalence between $TC(A)$ and $K(A)$ after p-adic completion.

The purpose of this paper is to give a brief introduction to the trace method and its topological version. After a review of background of K-theory in Chapter 2, We begin in Chapter 3 by discussing the Hochschild homology and the Dennis trace map. In particular, We give the concrete construction of the negative cyclic homology and the cyclic homology from the double complex $T_n^{\alpha,\beta}$ functor; see Definition 3.13. Based on Hochschild homology, we describe its topological analogue in Chapter 4. In Chapter 5, we set up the categories of p-cyclotomic spectra and cyclotomic spectra from the orthogonal S^1-spectra in the model structure with $\mathcal{G}$-equivalences as weak equivalences. Using these concepts, we provide a concise construction of the topological cyclic homology and connect it to the topological trace method.

The paper assumes familiarity with ordinary homotopy theory and category theory, including the model category and the geometric realization.

2. Background in algebraic K-theory

Unless stated otherwise, we always assume A to be a ring (not necessarily commutative) for this section.

Definition 2.1. A **projective module** over A is an A-module P with the property: given a surjective morphism of A-modules $M \xrightarrow{\phi} N$. For any A-module morphism $P \xrightarrow{\psi} N$, we have $\theta : P \to M$ such that the diagram commutes

$$\begin{array}{ccc} & & P \\ & \overset{\theta}{\swarrow} & \downarrow \psi \\ M & \xrightarrow{\phi} & N \end{array}$$

There is a theorem to characterize the projective modules.

Theorem 2.2. *An A-module is projective if and only if it is isomorphic to a direct summand in a free A-module. It is finitely generated and projective if and only if it is isomorphic to a direct summand in A^n for some n.*

Now we are ready to define the zeroth K-theory group of a ring A, denoted by $K_0(A)$. The isomorphism classes of finitely generated projective modules over A form an abelian monoid under the addition $\oplus$ and the identity element (the 0-module). Denote such monoid by $P(A)$. To see the well-definedness of this definition, there are a few things to check. Firstly, let P, Q be two finitely generated projective modules over A. If $P \cong P'$ and $Q \cong Q'$, we have $P \oplus P' \cong Q \oplus Q'$. This shows $\oplus$ is well-defined. Secondly, one can check $\oplus$ is commutative and associative once we pass to isomorphism classes.

Given any abelian monoid $(S,+)$, consider the equivalence relation in the Cartesian product $S \times S$

$$(a_1,b_1) \sim (a_2,b_2) \Longleftrightarrow a_1+b_2+c = a_2+b_1+c$$

for some $c \in S$. Write $G(S) \coloneqq S \times S/\sim$. This is called the **Grothendieck completion** of S.

Definition 2.3. The **zeroth K-theory group** of a ring A is defined as $K_0(A) \coloneqq G(P(A))$.

Example 2.4. Let k be a field. Then $P(k) \cong \mathbb{N}$ and $K_0(k) = \mathbb{Z}$.

In general cases, we want a well-defined K-theory of A consisting of a sequence of groups $K_n(A)$, for $n \geq 0$. Quillen in [4] gave one construction, which was done in his work at the Institute for Advanced Study during the year 1969-70. Let G be a group. There exists a contractible Hausdorff G-space with X/G paracompact, such that G acts freely and properly discontinuously on X (i.e. for all $x \in X$, there exists a neighborhood U_x of x such that $gU_x \cap U_x \neq \varnothing$ iff $g = 1$). In fact, such space is unique up to homotopy equivalence. Define the **classifying space** for G to be $BG \coloneqq X/G$, whose only non-vanishing homotopy group is its fundamental group, with $\pi_1 BG \cong G$. Write EG for X in the preceding setting.

Definition 2.5 (Quillen, [4])**.** For $n \geq 1$, we define the n^{th} **K-theory group** of A to be $K_n(A) \coloneqq \pi_n(BGL(A)^+)$, where B is the classifying space functor and $GL(A) \coloneqq GL_\infty(A)$ is the general linear group of infinite rank. X^+ is the Quillen's "+"-construction for X. We refer the reader to [7, Chap. 5] for more details.

Remark 2.6. We are interested in the following properties possessed by the Quillen's "+"–construction X^+ for X:

(1) $\pi_1(X,*) \to \pi_1(X^+,*)$ is the quotient map $\pi_1(X,*) \to \pi_1(X,*)/\pi$, where π is a perfect normal subgroup of $\pi_1(X^+,*)$.
(2) (X^+,X) is homologically acyclic, i.e. the homology $H_*(X^+,X;M)$ is 0 for an $\pi_1(X^+,*)/\pi$-module M.

Example 2.7. We have

$$\pi_1(BGL(A)^+)^{\text{ab}} = \pi_1(BGL(A))/E(A) = GL(A)/E(A) = GL(A)^{\text{ab}} = K_1(A),$$

where $(-)^{\text{ab}}$ denotes the abelianization, and $E(A)$ denotes the commutator subgroup of A. This is actually consistent with Whitehead's definition of K_1 in 1950's (see [6]).

Example 2.8 (Quillen, [5])**.** Let q be a prime. Then we have

$$K_n(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & , n=0 \\ \mathbb{Z}/(q^i-1) & , n = 2i-1 \\ 0 & , n=2i \end{cases}.$$

K-theory has closely connected to other fields, such as number theory. Kurihara showed that Vandiver's conjecture is equivalent to $K_n(\mathbb{Z}) = 0$ whenever $4 \mid n$. However, the case $K_n(\mathbb{Z})$ remains to be an open problem. To deal with the computation in general, Goodwillie came up with the **Dennis trace map** in [1],

$$D : K_q(A) \to HH_q(A).$$

We can factor this map through the negative cyclic homology $HC_q^-(A)$,

$$D : K_q(A) \to HC_q^-(A) \to HH_q(A).$$

Using the map D, one can approximate to the K-theory groups effectively. Before getting involved in the details, we first introduce the Hochschild homology.

3. Hochschild homology and trace method

Unless otherwise stated, we always assume R is a commutative and unital ring, and A is an associative and commutative R-algebra. We use the notation $A^{\otimes_R n}$ to denote A tensoring itself over R for n times. If $R = \mathbb{Z}$, we drop it from the notation. Hochschild gave his classical definition of Hochschild homology as follows.

Definition 3.1. The **Hochschild complex** $HH(A)$ is a chain complex defined as

$$HH(A) = \left(\cdots A^{\otimes 3} \xrightarrow{d} A^{\otimes 2} \xrightarrow{d} A \right),$$

with the differential d given by

$$d(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} + \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n.$$

The **Hochschild homology** $HH_*(A)$ is then defined as the homology of the Hochschild complex.

Definition 3.2. The **relative Hochschild complex** is defined to be

$$HH(A) = \left(\cdots A^{\otimes_R 3} \xrightarrow{d} A^{\otimes_R 2} \xrightarrow{d} A \right),$$

with d the same as before. The **relative Hochschild homology** $HH_*^R(A)$ is then defined as the homology of the relative Hochschild complex.

Example 3.3. Since we have assumed A is commutative, one can observe that when $n = 1$,

$$\begin{aligned} d(a_0 \otimes a_1) &= (-1)^1 a_1 a_0 + (-1)^0 a_0 a_1 \\ &= -a_1 a_0 + a_0 a_1 = 0. \end{aligned}$$

This implies $HH_0(A) = A$.

Example 3.4. Let $A = \mathbb{Q}$, then $A^{\otimes n} = \mathbb{Q}$. When n is odd, the differential d is the sum of equal number of pairs of -1 and $+1$, yielding $d = 0$. When n is even, we have an extra term $+1$, which in turn gives $d = 1$. Hence the differential d alternates between 0 and 1. Thus,

$$HH_n(A) = \begin{cases} \mathbb{Q} & , n = 0 \\ 0 & , n \neq 0. \end{cases}$$

Example 3.5. Let S be a commutative ring, we have computed $HH_0(S) = S$ in Example 3.3. In dimension 1,

$$HH_1(S) = \frac{\ker(S^{\otimes 2} \to S)}{\operatorname{im}(S^{\otimes 3} \to S^{\otimes 2})} \cong \Omega^1_{S/\mathbb{Z}},$$

where $\Omega^1_{S/\mathbb{Z}}$ is the Kähler differential.

We give another description of the Hochschild complex through cyclic bar construction. Let $\mathcal{C}$ be an abelian category and $X_\bullet$ be a simplicial object in $\mathcal{C}$. One always has an associated chain complex $s(X_\bullet)$, given by

$$\cdots \to X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0 \to 0 \to \cdots .$$

The differential is $d = \sum_i (-1)^i d_i$, where d_i are the face maps of $X_\bullet$. This construction is functorial, i.e. it induces an functor

$$s : \mathrm{Simp}(\mathcal{C}) \to \mathrm{Ch}_{\geq 0}(\mathcal{C}),$$

where $\mathrm{Simp}(\mathcal{C})$ is the category of simplicial objects in $\mathcal{C}$, and $\mathrm{Ch}_{\geq 0}(\mathcal{C})$ is the category of chain complex of non-negative degree. If $X_\bullet$ is a simplicial abelian group, then it gives rise to another chain complex $N(X_\bullet)$, which is the quotient of $s(X_\bullet)$ by the subcomplex generated by degenerate simplices. One can check the homology groups of $N(X_\bullet)$ and $s(X_\bullet)$ are naturally isomorphic to homotopy groups of $X_\bullet$. There is also a natural transformation $t : s \to N$. See [16, VII.5.2].

Definition 3.6. A **cyclic object** in a category $\mathcal{C}$ is a simplicial object $X_\bullet$ in $\mathcal{C}$ such that each X_n is equipped with an automorphism t_n of order $n+1$, and these are required to satisfy:

$$t_n d_i = \begin{cases} d_{i-1} t_{n-1} & , 1 \leq i \leq n \\ d_n & , i = 0 \end{cases}$$

$$t_n s_i = \begin{cases} d_{i-1} s_{n-1} & , 1 \leq i \leq n \\ s_n t_{n+1}^2 & , i = 0. \end{cases}$$

Here d_i, s_i are face maps and degeneracy maps, respectively.

Proposition 3.7. *The geometric realization $|X_\bullet|$ of a cyclic space $X_\bullet$ naturally admits an S^1-action.*

Proof. See [9, Chap. 4]. □

Let $B^{cy}_*(A)$ be the **cyclic bar construction** of A. For each index $*$, $B^{cy}_*(A) = A^{\otimes * + 1}$ is a simplicial ring. We can describe it through the diagram

$$A \begin{array}{c} \xleftarrow{d} \\ \xrightarrow{s} \\ \xleftarrow[d]{} \end{array} A \otimes A \begin{array}{c} \xleftarrow{d} \\ \xrightarrow{s} \\ \xleftarrow{d} \\ \xrightarrow{s} \\ \xleftarrow[d]{} \end{array} A \otimes A \otimes A \cdots .$$

The face maps are defined by

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & , 0 \leq i \leq n \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & , i = n, \end{cases}$$

and the degeneracy maps are defined by

$$s_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

There is a cyclic operation t_n associated to $B^{cy}_*(A)$ given by

$$t_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

With these conditions in hand, we can give a chain complex

$$C_*(A) = \left(B^{cy}_*(A), \sum_i (-1)^i d_i \right). \tag{3.8}$$

The **Hochschild homology** is defined by taking the homology of the previous complex, i.e. $HH_n(A) = H_n(C_*(A))$.

Remark 3.9. By Dold-Kan correspondence, $H_n(C_*(A))$ is isomorphic to $\pi_n(|B_*^{cy}(A)|)$, the homotopy groups of the realization of $B_*^{cy}(A)$.

With the operation t_n, Kan showed that $B_*^{cy}(A)$ is a cyclic object. So $|B_*^{cy}(A)|$ has an S^1-action.

Remark 3.10. One can verify that the simplicial perspective and the algebraic perspective give isomorphic chain complexes, and therefore present two equivalent definitions of Hochschild homology, as long as A is a flat R-algebra.

Lemma 3.11. *For any commutative ring A, $HH_*(A)$ admits the structure of a graded commutative ring in which all odd elements square to zero.*

Sketch of proof. In fact, $HH(A)$ has the structure of a commutative differential graded algebra by the Eilenberg-Zilber map. Thus from Künneth theorem, the homology $HH_*(A)$ is graded commutative. It is generally true for commutative differential graded algebras induced from simplicial commutative rings that the squares of odd elements are zero. See [11, Lemma 2.3]. □

Example 3.12. Write $\Omega^n_{A/\mathbb{Z}} = \bigwedge^n \Omega^1_{A/\mathbb{Z}}$, the n^{th} exterior power of $\Omega^1_{A/\mathbb{Z}}$. Actually $\bigwedge^* \Omega^1_{A/\mathbb{Z}} = \bigoplus_{n\geq 0} \bigwedge^n \Omega^1_{A/\mathbb{Z}}$ forms an exterior algebra. In general, we can relate n-dimensional Hochschild homology to the Kähler n-form. Namely, by Hochschild-Kostant-Rosenberg theorem, the above isomorphism extends to a graded ring isomorphism

$$HH_*(A) \xrightarrow{\simeq} \Omega^*_{A/\mathbb{Z}},$$

provided that A is a smooth algebra over $\mathbb{Z}$ (or more generally a localization of $\mathbb{Z}$). See [8, Chap. 9] for details.

In the light of the preceding setting, we are ready to introduce the trace method. To clarify our arguments, we first need to introduce negative cyclic homology and cyclic homology. We refer the reader to [1] for more concrete descriptions.

Define

$$\overline{B}_{p,q}(A) = \begin{cases} \overline{A}_{q-p} & , \ -\infty < p \leq q < \infty, \\ 0 & , \ p > q, \end{cases}$$

where $\overline{A}_n = A^{\otimes n}/\text{degeneracies}$. The vertical boundary b is given by

$$b = \sum_{i=0}^{q-p} (-1)^i d_i,$$

where d_i are face maps w.r.t. $\overline{A}_{q-p}$. The horizontal boundary B w.r.t. $\overline{A}_n$ ($n = q-p$) is given by

$$B = t_{n+2} s_n \sum_{k=1}^{n+1} ((-1)^n t_{n+1})^k.$$

Now for $-\infty \leq \alpha \leq \beta \leq \infty$, let $T_*^{\alpha,\beta}$ be the chain complex obtained from $\overline{B}_{*,*}$. Namely,

$$T_n^{\alpha,\beta}(A) = \prod_{\alpha \leq k \leq \beta} \overline{B}_{k,n-k}(A).$$

One can check if $\alpha = \beta = 0$, then $T_*^{0,0}(A)$ with vertical boundary b is the quotient of the Hochschild complex $C_*(A)$ in (3.8) by the subcomplex generated by the degenerate simplices. So $HH_n(A) = H_n T_*^{0,0}(A)$.

Definition 3.13. The **negative cyclic homology** of A is defined by

$$HC_*^-(A) \coloneqq H_n T_*^{-\infty,0}(A),$$

and the **cyclic homology** of A is defined by

$$HC_*(A) \coloneqq H_n T_*^{0,\infty}(A),$$

and the **periodic cyclic homology** of A is defined by

$$HP_*(A) \coloneqq H_n T_*^{-\infty,\infty}(A).$$

One can verify that there is a diagram of chain complexes with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_*^{-\infty,0}(A) & \longrightarrow & T_*^{-\infty,\infty}(A) & \longrightarrow & T_*^{1,\infty}(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\pi} & & \downarrow & & \downarrow{\scriptstyle=} & & \\ 0 & \longrightarrow & T_*^{0,0}(A) & \longrightarrow & T_*^{0,\infty}(A) & \longrightarrow & T_*^{1,\infty}(A) & \longrightarrow & 0 \end{array}$$

From the fact $T_*^{1,\infty}(A) \cong T_{*-2}^{0,\infty}(A)$ we deduce a map of long exact sequences
(3.14)

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & HC_{*-1}(A) & \longrightarrow & HC_*^-(A) & \longrightarrow & HP_*(A) & \longrightarrow & HC_{*-2}(A) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle\pi} & & \downarrow & & \downarrow{\scriptstyle=} & & \\ \cdots & \longrightarrow & HC_{*-1}(A) & \longrightarrow & HH_*(A) & \longrightarrow & HC_*(A) & \longrightarrow & HC_{*-2}(A) & \longrightarrow & \cdots \end{array}$$

For any group G, there is a natural injection from the free abelian group $\mathbb{Z}BG_\bullet$ generated by BG to the cyclic bar construction $B_*^{cy}(\mathbb{Z}G)$ of the group ring $\mathbb{Z}G$. Explicitly, on the standard basis for $\mathbb{Z}BG_k$, i.e. $BG_k = G^k$, it is given by

$$(g_1, \cdots, g_k) \mapsto (g_k^{-1} \cdots g_1^{-1}) \otimes g_1 \otimes \cdots \otimes g_k.$$

Apply the functor s and the natural transformation t, one gets a chain map

$$s(\mathbb{Z}BG_\bullet) \to s(B_*^{cy}(\mathbb{Z}G)) \xrightarrow{t} N(B_*^{cy}(\mathbb{Z}G)) = T_*^{0,0}(\mathbb{Z}G).$$

Now for A an associative and commutative R-algebra, define a chain map

$$T_*^{0,0}\mathbb{Z}GL_n A \xrightarrow{\varepsilon_n} T_*^{0,0}A.$$

This is given by the composition of $\mathbb{Z}GL_n A = \mathbb{Z}GL_1 M_n(A) \to M_n(A)$ and the "trace" $T_*^{0,0}M_n(A) \to T_*^{0,0}A$. Iterating the inclusions $GL_n(A) \hookrightarrow GL_{n+1}(A)$ yields a chain map

$$T_*^{0,0}\mathbb{Z}GL(A) \xrightarrow{\varepsilon} T_*^{0,0}A.$$

Taking the homology functor $H_*(-)$ to both sides gives

$$H_n T_*^{0,0}\mathbb{Z}GL(A) \to \pi_n(BGL(A)^+) \cong K_n(A) \xrightarrow{\tau} HH_n(A). \tag{3.15}$$

Definition 3.16. τ defined in (3.15) is called the **Dennis trace map**.

Theorem 3.17. *For any associative and commutative R-algebra A, there is a natural map α making the diagram commute*

$$\begin{array}{ccc} & & HC_n^-(A) \\ & \overset{\alpha}{\nearrow} & \downarrow \pi \\ K_n(A) & \xrightarrow{\tau} & HH_n(A) \end{array}$$

where π is as in (3.14) and τ is the Dennis trace map.

Remark 3.18. τ and α can also be construct in the relative case. Let $f : B \to A$ be a map of R-algebras, then there are natural maps

$$\begin{array}{ccc} & & HC_n^-(f) \\ & \overset{\alpha}{\nearrow} & \downarrow \pi \\ K_n(f) & \xrightarrow{\tau} & HH_n(f) \end{array}$$

making the square commute (up to sign) in

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & K_n(A) & \longrightarrow & K_n(f) & \longrightarrow & K_{n-1}(B) & \longrightarrow & \cdots \\ & & \downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \\ \cdots & \longrightarrow & HH_n(A) & \longrightarrow & HH_n(f) & \longrightarrow & HH_{n-1}(B) & \longrightarrow & \cdots \end{array}$$

and similar for τ.

We can have a better approximation to the algebraic K-theory via the following theorem.

Theorem 3.19 (Goodwillie, [1])**.** *Let $f : B \to A$ be a nilpotent extension, i.e. f is a monomorphism with a nilpotent kernel. Then the following square is homotopy cartesian:*

$$\begin{array}{ccc} K(B) \otimes \mathbb{Q} & \xrightarrow{\alpha_{\mathbb{Q}}(f)} & HC^-(B \otimes \mathbb{Q}) \\ \downarrow & & \downarrow \\ K(A) \otimes \mathbb{Q} & \xrightarrow{\alpha_{\mathbb{Q}}(f)} & HC^-(A \otimes \mathbb{Q}). \end{array}$$

where $\alpha_{\mathbb{Q}}(f)$ is as in Theorem 3.17 and Remark 3.18. Moreover, we have

$$K_n(f) \otimes \mathbb{Q} \cong HC_n^-(f \otimes \mathbb{Q}).$$

For cyclic homology version, we have

$$K_n(f) \otimes \mathbb{Q} \cong HC_{n-1}(f) \otimes \mathbb{Q}.$$

Goodwillie's result still fails to provide a computable scheme of K-theory in p-adic version. For that matter, we will move to the **topological Hochschild homology**, or **THH** in abbreviation, in the next section.

4. Topological Hochschild homology

Topological Hochschild homology (THH) was first proposed by Bökstedt in 1990, which reveals an analog to the Hochschild homology. It has many equivalent definitions now, and we shall only focus on one in this paper. We begin with the concepts of spectra.

4.1. Spectra.

Definition 4.1. A **prespectrum** T is a sequence of based spaces T_n, $n \geq 0$, and based maps $\sigma_n : \Sigma T_n \to T_{n+1}$. A map $T \to V$ between prespectra is a sequence of maps $T_n \to V_n$ that commute with the structure maps σ_n.

Definition 4.2. An **Ω-spectrum** is a prespectrum such that the adjoints $\tilde{\sigma}_n : T_n \to \Omega T_{n+1}$ of the structure maps are weak homotopy equivalences.

Definition 4.3. A **spectrum** is an **Ω-spectrum** $\{T_n\}_{n\geq 0}$ such that the adjoints $\tilde{\sigma}_n : T_n \to \Omega T_{n+1}$ of the structure maps are homeomorphisms.

Example 4.4 (Eilenberg-MacLane spectrum)**.** Recall that in homotopy theory, the Eilenberg-MacLane spaces $K(A, n)$ have the homotopy type of CW complexes. For each choice of A and n, the space $K(A, n)$ is unique up to homotopy equivalence. Milnor theorem shows that if X has the homotopy types of a CW complex, then so does ΩX. So by Whitehead theorem, we have a homotopy equivalence

$$\tilde{\sigma}_n : K(A, n) \to \Omega K(A, n + 1).$$

This map has an adjoint

$$\sigma_n : \Sigma K(A, n) \to K(A, n + 1).$$

So $\{K(A, n)\}_{n\geq 0}$ is a prespectrum, called the **Eilenberg-MacLane prespectrum**. Using the "spectrification", we can obtain the corresponding **Eilenberg-MacLane spectrum**. See Section 25.7 of [10] for details.

Example 4.5 (Suspension and sphere spectrum)**.** Let X be a based space. It gives rise to a **suspension prespectrum** $\Sigma^\infty X$ via

$$(\Sigma^\infty X)_n = X \wedge S^n.$$

The structure map is given by $\Sigma(\Sigma^n X) = \Sigma^{n+1} X$. We use "spectrification" to $\Sigma^\infty X$ by letting $Y_i = \mathrm{colim}_{j\geq 0} \Omega^j (\Sigma^\infty X)_{i+j}$. Y is then a spectrum, called the **suspension spectrum**. When $X = S^0$, this is called the **sphere spectrum**. See Section 25.7 of [10] for details.

Example 4.6 (Orthogonal spectrum)**.** An orthogonal spectrum consists of:

(1) a sequence of based spaces $\{X_n\}_{n\geq 0}$ with structure maps $\sigma_n : X_n \wedge S^1 \to X_{n+1}$,
(2) a basepoint-preserving continuous left action of the orthogonal group $O(n)$ on X_n for each $n \geq 0$.

This data is subject to the fact that, for each $m, n \geq 0$, we have an $O(n) \times O(m)$-equivariant map

$$\sigma^m : X_n \wedge S^m \to X_{n+m},$$

which is the composite of

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge S^{m-1}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge S^{m-2}} \cdots \xrightarrow{\sigma_{n+m-1}} X_{n+m}.$$

Here the orthogonal group $O(m)$ acts on S^m, and $O(n) \times O(m)$ acts on X_{n+m} by restriction, along orthogonal sum, of the $O(n + m)$-action. A map $f : X \to Y$ between orthogonal spectra is compatible with the structure maps in the sense that $f_{n+1} \circ (\sigma_n \circ (- \wedge S^1)) = \sigma_n \circ (f_n \wedge S^1)$ for all $n \geq 0$.

Definition 4.7. If X is a prespectrum, $n \in \mathbb{Z}$, then the n^{th} **homotopy group** of X is

$$\pi_n(X) \coloneqq \operatorname{colim}_i \pi_{n+i} X_i.$$

A π_***-isomorphism** of prespectra is a sequence of maps inducing isomorphisms at the level of every homotopy group.

4.2. **Construction of topological Hochschild homology.** We are ready to establish the topological Hochschild homology. By the work of Elmendorf, Kriz, Mandell, and May ([13, Chap. VII]), there is a category $\mathcal{M}_S$ of spectra that is symmetric monoidal with the operation to be the smash product $\wedge$. The unit object is equivalent to the sphere spectrum S. The monoids in this category are called S-algebras. For any commutative S-algebra R, there is a symmetric monoidal category of R-modules. It naturally derives the construction of an R-algebra A. According to [13], we shall have the following analogues:

(4.8)

ring	ring spectrum
$\otimes$	$\wedge$
$\mathbb{Z}$	S (sphere spectrum)
A	HA (Eilenberg-MacLane spectrum)
algebra over rings	algebra over ring spectra
$B^{cy}_{*,\otimes}$	$B^{cy}_{*,\wedge}$
$HH(A)$	$THH(A)$

By substituting the terms in cyclic bar construction $B^{cy}_*(A)$ according to the table (4.8), we can describe the topological Hochschild homology in terms of the brave new algebra version of the Hochschild complex. Write $A^{\wedge n}$ for the n-fold $\wedge$-power of A, where the wedge is over R. Let

$$\phi : A \wedge A \to A \quad \text{and} \quad \eta : R \to A$$

be the product and unit of A. Let

$$t : A \wedge A^{\wedge n} \to A^{\wedge n} \wedge A$$

be cyclic permutation isomorphisms. The topological analogue of passage from a simplicial R-module to a chain complex of R-modules is passage from a simplicial spectrum to its geometric realization.

Definition 4.9. Let R be a commutative S-algebra and A be an commutative R-algebra. Let $B^{cy}_{*,\wedge}$ be the cyclic bar construction of A. For each index $*$, $B^{cy}_{*,\wedge} = A^{\wedge * + 1}$. The diagram is

$$A \;\underset{d}{\overset{d}{\leftleftarrows}}\!\!\xrightarrow{s}\; A \wedge A \;\;\begin{smallmatrix}\xleftarrow{d}\\ \xrightarrow{s}\\ \xleftarrow{d}\\ \xrightarrow{s}\\ \xleftarrow{d}\end{smallmatrix}\;\; A \wedge A \wedge A \cdots .$$

The face maps d_i and the degeneracy maps s_i of $B^{cy}_{n,\wedge}$ are given by

$$d_i = \begin{cases} \phi \wedge (\mathrm{id})^{n-1} & , i = 0 \\ \mathrm{id} \wedge (\mathrm{id})^{i-1} \wedge \phi \wedge (\mathrm{id})^{n-i-1} & , 0 < i < n \\ (\phi \wedge (\mathrm{id})^{n-1}) \circ t & , i = n \end{cases}$$

$$s_i = \mathrm{id} \wedge (\mathrm{id})^i \wedge \eta \wedge (\mathrm{id})^{n-i}.$$

The **relative topological Hochschild homology** $THH^R(A)$ of A is defined to be $THH^R(A) \coloneqq |B^{cy}_{*,\wedge}|$. If $R = S$ is a sphere spectrum, then the relative topological Hochschild homology becomes the **(absolute) topological Hochschild homology** $THH(A) \coloneqq THH^S(A)$. The homotopy groups of this spectrum are denoted by $THH^R_n(A) \coloneqq \pi_n THH^R(A)$, called the **topological Hochschild homology groups of** A.

Notation 4.10. If A is an ordinary ring, we define its topological Hochschild homology by

$$THH(A) \coloneqq THH^S(HA),$$

where HA is the Eilenberg-MacLane spectrum of A.

Remark 4.11. In fact, the construction in Definition 4.9 can be generalized to any symmetric monoidal ∞-category. See [11, Page 9-10].

Remark 4.12. There is another algebraic definition of topological Hochschild homology in derived categories via the enveloping R-algebra $A^e = A \wedge_R A^{\mathrm{op}}$. Namely, let M be an (A, A)-bimodule. We define $THH^R(A; M) \coloneqq M \wedge_{A^e} A$. On passage to homotopy groups, we define $THH^R_*(A; M) = \mathrm{Tor}^{A^e}_*(M, A)$. When $M = A$, we drop it from the notation. This $THH^R(A)$ does **not** totally agree with the one defined stated in Definition 4.9, but they are weakly equivalent. See [13, Chap. IX, Propsition 2.5] for a reference.

There is an efficient way to compute the topological Hochschild homology. We shall state it as a theorem without explanations.

Theorem 4.13. *Let E be a commutative ring spectrum and A be a commutative R-algebra. If $E_*(R)$ is a flat R_*-module, or if E is a commutative S-algebra, then there is a spectral sequence with*

$$E^2_{p,q} = HH^{E_*(R)}_{p,q}(E_*(A)) \Longrightarrow E_{p+q}(THH^R(A)).$$

Proof. See [13, Proposition IX.1.11]. □

The similarity of constructions in topological Hochschild homology (Definition 4.9) and the one in Hochschild homology has motivated us to seek a connection between them. Luckily, we have one in low degree.

Proposition 4.14. *For an associative, commutative and unital ring A, the map*

$$THH_*(A) \to HH_*(A)$$

is 3-connected, i.e. an isomorphism in degree $* \leq 2$ and subjective in degree 3.

Proof. See [11, Proposition 3.11]. □

Example 4.15. For A a $\mathbb{Q}$-algebra, HA is rational i.e. the map $HA \to H\mathbb{Q} \wedge HA$ is an equivalence. Moreover, this equivalence is symmetric monoidal and identifies the smash product on the spectra side with the tensor product on the chain complexes side. So the topological Hochschild homology of such an Eilenberg-MacLane spectrum coincides with its Hochschild homology when considered as a differential graded algebra over $\mathbb{Q}$. Hence the map $THH_*(A) \to HH_*(A)$ is an isomorphism.

Example 4.16. For sphere spectrum S, we have $THH(S) = S$.

Theorem 4.17. *We have the following homotopy groups for all $\mathbb{Z}/p^n$ except for $\mathbb{Z}/4$:*

$$THH_*(\mathbb{Z}/p^n) = \begin{cases} \bigoplus_{i=0}^{k} \mathbb{Z}/\gcd(i,p^n) & , * = 2k \\ \bigoplus_{i=1}^{k} \mathbb{Z}/\gcd(i,p^n) & , * = 2k-1 \\ 0 & , * < 0 \end{cases}$$

$$THH_*(\mathbb{Z}) = \begin{cases} \mathbb{Z}/k & , * = 2k-1 \\ \mathbb{Z} & , * = 0 \\ 0 & , \textit{else} \end{cases}$$

Proof. See [11, Corollary 4.2]. □

Theorem 4.17 is actually a corollary of the following results, which is for $n = 1, \infty$ due to Blumberg, Cohen and Schlichtkrull [14] and for other n due to Nitu Kitchloo [15]. We shall state it as follows.

Theorem 4.18. *We have the following equivalences as $\mathbb{E}_1$-ring spectra:*

(1) $THH(\mathbb{Z}/p) \simeq H\mathbb{F}_q \wedge \Omega S^3$ for any p;
(2) $THH(\mathbb{Z}/p^n) \simeq H\mathbb{Z}/p^n \wedge \mathrm{fib}(\Omega S^3 \to K(\mathbb{Z}/p^{n-1}, 2))$ for odd p;
(3) $THH(\mathbb{Z}/2^n) \simeq H\mathbb{Z}/p^n \wedge \mathrm{fib}(\Omega S^3 \to K(\mathbb{Z}/p^{n-2}, 2))$ for $p = 2$ and $n \geq 3$;
(4) $THH(\mathbb{Z}) \simeq H\mathbb{Z} \wedge \tau_{\geq 3}\Omega S^3$.

Example 4.19 (Bökstedt)**.** $THH_*(\mathbb{F}_p) = \mathbb{F}_p[x]$, with $|x| = 2$. Generally, for any perfect field k with characteristic p, we have $THH_*(k) = k[x]$ with $|x| = 2$.

Proof. This immediately follows from Theorem 4.18 since the homology of ΩS^3 is polynomial on a degree 2 generator. □

5. A glimpse into topological trace method

Bökstedt-Hsiang-Madsen [2] constructed a topological version of the Dennis trace map: $D' : K(A) \to THH(A)$. Subsequently, they gave the construction of topological cyclic homology TC, as well as the cyclotomic trace map to TC. This lifted the topological Dennis trace

$$K(A) \to TC(A) \to THH(A).$$

After that, McCarthy's work [3] showed it has a good approximation to the algebraic K-theory after p-completion. To introduce the idea of topological trace method, we first review some basic knowledge in equivariant stable homotopy theory.

5.1. Equivariant G-spectra.

We shall assume G is a finite compact Lie group for this section.

Definition 5.1. A **G-space** is a topological space with the group action $G \curvearrowright X$ such that $ex = x$ and $g(g'x) = (gg')x$ for any $x \in X$. A map $f : X \to Y$ between G-spaces is a **G-map** if $gf(x) = f(gx)$ for any $g \in G$. The category of G-maps and G-spaces forms a category, denoted by GTop.

For $H \subset G$, the **H-fixed points set** is $X^H = \{x : hx = x \ \forall h \in H\}$. Define the Weyl groups NH to be the normalizer of H in G, $WH = NH/H$. It is clear X^H is a WH-space. In the equivariant theory, the orbit G/H plays the role of points, and the set of G-maps from G/H to itself can be identified with the group WH.

Definition 5.2. Let X be an H-space, $H \subset G$. There is an induced G-space $G \times_H X$ obtained by G acting diagonally on the product. That is, identifying (gh, x) with (g, hx) for any $g \in G$, $h \in H$ and $x \in X$ on the space $G \times X$. The **homotopy orbit space** of X under G is defined to be $X_{hG} \coloneqq EG \times_G X$.

Dually, we are able to define the function space $F_H(G, X)$ according to the adjunction $\mathrm{GTop}(G \times_H X, Y) \cong \mathrm{GTop}(X, F_H(G, Y))$. Here $F_H(G, Y)$ is the space of H-maps from G to Y with left action by G induced by right action of G on itself, $(g \cdot f)(g') = f(g'g)$. Let $F(G, X)$ be the function space consisting of G-maps from G to X. Then $F_H(G, X)$ is a subspace of $F(G, X)$. There is a homeomorphism $F_H(G, X) \cong F(G/H, X)$.

Definition 5.3. A **G-universe** U is a countably infinite dimensional inner product space with an G-action through isometries. U is said to be **complete** if it contains every irreducible representation of G (up to isomorphism).

Remark 5.4. U can be written as a direct sum of subspaces $(V_i)^\infty$, where $\{V_i\}$ runs over a set of distinct irreducible representations of G.

Similar to non-equivariant ones, the equivariant spectra can be defined as follows.

Definition 5.5. A **G-prespectrum** X indexed in U is the data of a family of based G-spaces $X(V)$ for every finite-dimensional subspace $V \subset U$ and G-maps $\sigma_{V,W} : S^W \wedge X(V) \to X(W \oplus V)$ for each pair $V, W \subset U$. We require that these structure maps satisfy the following conditions:

(1) $\sigma_{V,I} = \mathrm{id}$, where I is the trivial representation.
(2) For any $V, W, Z \subset U$, we have the commutative diagram

$$\begin{array}{ccc} S^Z \wedge S^W \wedge X(V) & \xrightarrow{S^Z \wedge \sigma_{V,W}} & S^Z \wedge X(W \oplus V) \\ \downarrow \simeq & & \downarrow \sigma_{W \oplus V, Z} \\ S^{Z \oplus W} \wedge X(V) & \xrightarrow{\sigma_{V, Z \oplus W}} & X(Z \oplus W \oplus V). \end{array}$$

A G-prespectrum is an **G-Ω-spectrum** if for all V and W, the adjoint to the structure map $\widetilde{\sigma}_{V,W} : X(V) \xrightarrow{\simeq} \Omega^W X(W \oplus V)$ is a weak homotopy equivalence.

Definition 5.6. The n^{th} **homotopy group** of a G-prespectrum is

$$\pi_n^H(X) \coloneqq \mathrm{colim}_V \pi_n^H \Omega^V X(V),$$

where $n \geq 0$.

Definition 5.7. A G-prespectrum is a **G-spectrum** if for all V and W, the structure maps $\sigma_{V,W}$ are homeomorphisms.

Example 5.8 (Orthogonal G-spectrum)**.** Following [18, Sec. 2], let $\mathscr{I}(\mathbb{R}^n, V)$ be the space of linear isometries from $\mathbb{R}^n$ to V, where V is a G-representation of a G-universe U of finite dimension n. The orthogonal group $O(n)$ acts transitively on $\mathscr{I}(\mathbb{R}^n, V)$ by pre-composition. Let X be an orthogonal spectrum defined in Example 4.6. Define $X(V) = \mathscr{I}(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$. If $V = \mathbb{R}^n$, there is a canonical homeomorphism

$$\begin{aligned} X_n &\to X(\mathbb{R}^n), \\ x &\mapsto (\mathrm{id}, x). \end{aligned}$$

In general, for any V and $\varphi \in \mathscr{I}(\mathbb{R}^n, V)$, we have

$$X_n \to X(V),$$
$$x \mapsto (\varphi, x).$$

$X(V)$ becomes a G-space by the rule $g \cdot (\varphi, x) = (g\varphi, gx)$. In the preceding settings, $X(V)$ depends only on V, up to homeomorphism. To describe the structure maps $\sigma_{V,W} : S^W \wedge X(V) \to X(W \oplus V)$, we first set $m = \dim W$, $n = \dim V$ and choose $\psi \in \mathscr{I}(\mathbb{R}^n, W)$, $\varphi \in \mathscr{I}(\mathbb{R}^n, V)$. Then $\sigma_{V,W}$ is given through

$$\sigma_{V,W}(s \wedge (\varphi, x)) = (\varphi \wedge \psi, \sigma^m(x \wedge \psi^{-1}(s)))$$

in $X(W) = \mathscr{I}(\mathbb{R}^{n+m}, V \oplus W)_+ \wedge_{O(n+m)} X_{n+m}$, where $s \in S^W$. One can check $\sigma_{V,W}$ is well-defined and independent of the choice of ψ. It is also $O(V) \times O(W)$-equivariant. The associativity can also be easily verified according to the condition (2) Definition 5.5.

Suppose X is a G-spectrum, we write $X^H(V) = (X(V))^H$ for the **(categorical) fixed points**, where V is a G-representation fixed by H. Similarly to Definition 5.2, we can define the homotopy orbit spaces of G-spectra and the function spectra. However, the fixed-point functor of G-spectra carries a defect: the associativity of operators is not guaranteed. For example, $(X \wedge Y)^H \ncong X^H \wedge Y^H$. In case of suspension spectra, $(\Sigma^\infty Z)^H \ncong \Sigma^\infty(Z^H)$. So we ought to find a better model for the fixed points. This triggered the appearance of geometric fixed points.

Let $\mathcal{F}_H = \{K : K \leq G, K \text{ doesn't contain } H\}$ and $E\mathcal{F}_H$ be the universal $\mathcal{F}_H$-space. $E\mathcal{F}_H$ is characterized by

$$(E\mathcal{F}_H)^K \simeq \begin{cases} \text{contractible} & , K \in \mathcal{F}_H \\ \varnothing & , K \notin \mathcal{F}_H. \end{cases}$$

Example 5.9. Take $H = C_p \subset C_p = G$, then $\mathcal{F}_H = \{*\}$, yielding $E\mathcal{F}_H \simeq EC_p$.

Use the notation $E\mathcal{F}_{H+}$ to denote its disjoint union with a G-fixed basepoint. There is a cofiber sequence $E\mathcal{F}_{H+} \to S^0 \to \widetilde{E\mathcal{F}_H}$. We define the **geometric fixed points** of X at H to be $\Phi^H(X) := \left(\widetilde{E\mathcal{F}_H} \wedge X\right)^H$. Applying the functor $- \wedge X$ to the preceding cofiber sequence and then taking the fixed points at H, we obtain

$$(E\mathcal{F}_{H+} \wedge X)^H \to X^H \xrightarrow{\beta} \Phi^H(X). \tag{5.10}$$

The map β links fixed points to geometric fixed points.

5.2. **Topological cyclic homology.** To build the topological cyclic homology, we need the concept of cyclotomic spectrum. We shall follow [19, Chap. 4] to present a framework of construction. Fix a prime p and define ρ_p to be the p^{th} root isomorphism $S^1 \to S^1/C_p$. It induces a functor ρ_p^* from the category of S^1/C_p-spaces to the category of S^1-spaces.

Definition 5.11. A **p-precyclotomic spectrum** X consists of an orthogonal S^1-spectrum X together with a map of orthogonal S^1-spectra

$$t : \rho_p^* \Phi^{C_p} X \to X.$$

A morphism of p-precyclotomic spectra consists of a map of orthogonal S^1-spectra $X \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} \rho_p^*\Phi^{C_p}X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \rho_p^*\Phi^{C_p}Y & \longrightarrow & Y \end{array}$$

Proposition 5.12. *The category of p-precyclotomic spectra $\mathscr{S}_p^{precyc}$ is enriched over the category of orthogonal S^1-spectra. $\mathscr{S}_p^{precyc}$ is a complete category, with limits created in the category of orthogonal S^1-spectra.*

Definition 5.13. Let $\mathcal{G}$ be a family of subgroups of S^1 such that

(1) every group $G \in \mathcal{G}$, G is a closed subgroup of S^1,
(2) $\mathcal{G}$ is closed under taking closed subgroups and conjugations.

Let X, Y be two orthogonal S^1-spectra. An **$\mathcal{G}$-equivalence** is a map $f : X \to Y$ that induces isomorphisms on homotopy groups π_*^G for all subgroups G in $\mathcal{G}$.

The following theorem is crucial to construct the topological cyclic homology.

Theorem 5.14. *There is a stable, positive stable, and positive complete stable compactly generated model structure on the category of orthogonal S^1-spectra where the weak equivalences are the $\mathcal{G}$-equivalences.*

Definition 5.15. A **p-cyclotomic spectrum** X is a p-precyclotomic spectrum satisfying that the structure map t induces an $\mathcal{G}_p$-equivalence in the category $\mathscr{S}_p^{precyc}$

$$\rho_p^*\mathcal{L}\Phi^{C_p}X \to X,$$

where $\mathcal{L}\Phi^{C_p}$ is the left derived functor of the geometric fixed point functor Φ^{C_p}, and $\mathcal{G}_p$ is a collection of p-subgroups of S^1, $\mathcal{G}_p = \{C_{p^n}\}$.

Definition 5.16. A **precyclotomic spectrum** X consists of an orthogonal S^1-spectrum X together with the structure maps for all $n \geq 1$

$$t_n : \rho_n^*\Phi^{C_n}X \to X.$$

It is subject to the condition that for all $m, n \geq 1$, the following diagram commutes

$$\begin{array}{ccccc} \rho_{mn}^*\Phi^{C_{mn}}X & \longrightarrow & \rho_n^*\Phi^{C_n}(\rho_m^*\Phi^{C_m}X) & \xrightarrow{\rho_n^*\Phi^{C_n}t_m} & \rho_n^*\Phi^{C_n}X \\ \downarrow & & \downarrow & & \downarrow \\ \rho_m^*\Phi^{C_m}(\rho_n^*\Phi^{C_n}X) & \xrightarrow{\rho_m^*\Phi^{C_m}t_n} & \rho_m^*\Phi^{C_m}X & \xrightarrow{t_m} & X \end{array}$$

A morphism of precyclotomic spectra consists of a map of orthogonal S^1-spectra $X \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} \rho_n^*\Phi^{C_n}X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \rho_n^*\Phi^{C_n}Y & \longrightarrow & Y \end{array}$$

Definition 5.17. A **cyclotomic spectrum** X is a precyclotomic spectrum satisfying that the structure map t_n induces an $\mathcal{G}$-equivalence in the category of precyclotomic spectra

$$\rho_n^* \mathcal{L}\Phi^{C_n} X \to X,$$

where $\mathcal{L}\Phi^{C_n}$ is the left derived functor of the geometric fixed point functor Φ^{C_n}, and $\mathcal{G}$ is a collection of closed subgroups of S^1 agreeing with Definition 5.13.

Example 5.18. The S^1-equivariant sphere spectrum $\mathcal{S}$ has a canonical structure as a cyclotomic spectrum induced by the sphere spectrum S together with the canonical isomorphisms $\rho_n^* \Phi^{C_n} S \cong S$.

Theorem 5.19. *$THH(A)$ is a cyclotomic spectrum for an associative, commutative and unital ring A.*

The readers can refer [20, Theorem 5.2] for the detailed discussion on this theorem.

Definition 5.20. Let X be a cyclotomic spectrum. For every $n \geq 1$, the **restriction map** R_n is the map of non-equivariant orthogonal spectra

$$R_n : X^{C_{mn}} \cong (\rho_n^* X^{C_n})^{C_m} \xrightarrow{(\rho_n^*\beta)^{C_m}} (\rho_n^* \Phi^{C_n} X)^{C_m} \xrightarrow{t_n^{C_m}} X^{C_m}, \tag{5.21}$$

where β is defined in (5.10). The **Frobenius map** F_n is the map of non-equivariant orthogonal spectra induced by the inclusion of fixed points

$$F_n : X^{C_{mn}} \to X^{C_m}.$$

When $n = 1$, we urge $R_1 = F_1 = \mathrm{id}$.

It is straightforward to check the following facts.

Proposition 5.22. *R_n and F_n satisfy*

(1) $R_m R_n = R_{mn}$*,*
(2) $F_m F_n = F_{mn}$*,*
(3) $F_m R_n = R_n F_m$*, or* $F \circ R = R \circ F$*.*

we now explain how to construct the topological cyclic homology.

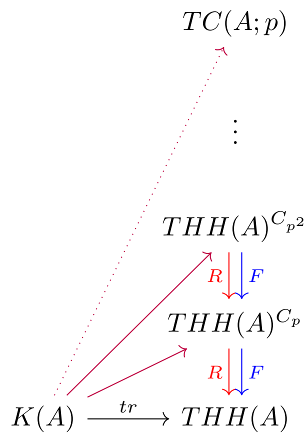
Definition 5.23. Let X be a cyclotomic spectrum that is fibrant in the model structure defined in Theorem 5.14. We define the **topological cyclic homology** of X to be

$$TC(X) \coloneqq \mathrm{holim}_{R,F} X^{C_{p^n}}.$$

Let Y be a p-cyclotomic spectrum that is fibrant in the model structure defined in Theorem 5.14. We define the **topological cyclic homology** of Y to be

$$TC(Y; p) \coloneqq \mathrm{holim}_{R,F} Y^{C_{p^n}}.$$

Remark 5.24. Set $X = THH(A)$ for an associative, commutative and unital ring A. By Theorem 5.19, we may realize $TC(A;p)$ through the following tower



Here $K(A)$ is the algebraic K-theory spectrum and "*tr*" is the cyclotomic trace. See [20, Sec. 5] for the concrete description.

The appearance of TC gives rise to an elegant way to recover the information of the K-theory groups $K_n(A)$. We have the following result to demonstrate this approach.

Theorem 5.25 (McCarthy, [3])**.** *Let $f : A \to A'$ be a nilpotent extension, i.e. a momomorphism of rings whose kernel I is a nilpotent ideal. Then the following square is homotopy cartesian after completion at p for any prime p:*

$$\begin{array}{ccc} K(A) & \xrightarrow{tr} & TC(A) \\ \downarrow & & \downarrow \\ K(A') & \xrightarrow{tr} & TC(A'). \end{array}$$

Moreover, the induced map of homotopy fibers $K(f) \to TC(f)$ becomes an equivalence after p-adic completion.

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