Riemann-Roch Theorem: An Observation

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Abstract

This paper gives a brief introduction to the Riemann-Roch Theorem in algebraic geometry. We will develop the appropriate tool to state it in a suitable way (without the use of sheaf language). Background of some basic commutative algebra is assumed.

- Kerr M. Lecture Notes: Algebraic Geometry III/IV.
- Vakil R. THE RISING SEA: Foundations of Algebraic Geometry.

1 Introduction

There is a classical problem arise from the study of Riemann surface: is there any non-constant meromorphic forms ω in a given Riemann surface M with genus g?

If we let $g \geq 1$, ω cannot have a simple pole at only one point $p \in M$. This is a direct corollary of residue theorem. Then how about ω have a double pole at p? If we let $g = 0$ or 1, it can happen since we have the Weierstrass $\wp$-function, however it depends on the point p when $g \geq 2$.

Is there a general way to solve this problem? Namely, can we guarantee such ω exists (but singular at one point p) for a given g? The answer is yes, and this is really a result of the well-known Riemann-Roch Theorem.

Theorem 1.1 (Riemann-Roch). *For every divisor D on a compact Riemann surface of genus g, we have*

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

In particular, we have for $f \in \mathcal{K}(M)$ with a single pole (at p) with $\nu_p(f) \geq -k$ has dimension $\geq \max\{1, k - g + 1\}$. Hence if we let the order of the pole at p ≥ 2, we can guarantee the existence of such ω as long as $k - g + 1 \geq 2$.

In order to have a clear picture of Riemann-Roch Theorem, we need to develop some tools. In the following sections, we will introduce some basic concepts needed and hopefully we will provide a simple proof towards it.

2 Basic concepts

Definition 2.1. *A **Riemann surface** M is a compact complex 1-manifold.*

We will use letter M to denote a Riemann surface unless stated otherwise.

Example 2.2. $\mathbb{P}^1$, *the (complex) projective space of dimension 1, is a Riemann surface, which is actually the (Riemann) sphere* $S^2 = \hat{\mathbb{C}}$.

Definition 2.3. *A meromorphic (resp. holomophic) function* $f \in \mathcal{K}(M)$ *(resp. Ø(M)) is a collection of continuous maps* $f_\alpha : U_\alpha \to \mathbb{P}^1$ *such that*

1. *the* $\{f_\alpha\}$ *agrees on overlaps,*
2. $f_\alpha \circ \phi_\alpha$ *is meromorphic (resp. holomorphic) function for all* α,

where $\{\phi_\alpha : U_\alpha \to \mathbb{C}\}$ *is the coordinate charts.*

Definition 2.4. *A* ***(complex) holomorphic (resp. meromorphic) 1-form*** $\omega \in \Omega^1(M)$ *(resp.* $\mathcal{K}(M)$*) on* M *is a collection of expressions* $\omega_\alpha = f_\alpha(z_\alpha)dz_\alpha$ *with* $f_\alpha : V_\alpha \to \mathbb{C}$ *holomorphic (resp. meromorphic) and satisfying* $\omega_\beta |_{V_{\alpha\beta}} = \Phi^*_{\alpha\beta}(\omega_\alpha |_{V_{\alpha\beta}})$ *for any* α, β *and* $\Phi^*_{\alpha\beta}$ *the pullback of the transition of charts* $\{z_\alpha\}$.

Example 2.5. *Let* $M = \mathbb{P}^1$, $\omega_1 = \omega$ *and* $\omega_2 = dz$. *Here* $z = \frac{Z_1}{Z_0}$ *on* $\mathbb{P}$ *as usual, and* dz *looks as if it should be not just meromorphic but holomorphic. But in the "coordinate at* ∞*",* $\omega = \frac{Z_0}{Z_1}$, dz *becomes* $d(\frac{1}{\omega}) = -\frac{d\omega}{\omega^2}$. *So* dz *in fact has a pole of order 2 at* $[0 : 1]$.

Now consider $F(z) = \frac{\omega_1}{\omega_2} = \frac{\omega}{dz} \in \mathcal{K}(\mathbb{P})$, *then we have* $\omega = F(z)dz$. *Hence we actually have*

$$\mathcal{K}(\mathbb{P}) = \left\{ \frac{P(z)}{Q(z)} dz \mid P \in \mathbb{C}[z], Q \in \mathbb{C}[z] \backslash \{0\} \right\}.$$

Remark 2.6. *In modern approach, the complex holomorphic 1-form is just the global section of* Ω^1, *where* $\Omega^1(U_i, \varphi_i)$ *is the free* $\mathcal{O}(U_i, \varphi_i)$*-module of rank 1 with basis element denoted by* dz_i, *and* Ω^1 *is defined to be the sheaf with* $\Omega^1(U_i) = \Omega^1(U_i, \varphi_i)$, *which is obtained by gluing compatible sheaves over* U_i.

Also, the complex meromorphic 1-form is the global section of the sheaf $\mathcal{M}^1 = \mathcal{M} \otimes_{\Omega^0} \Omega^1$. *Here* Ω^0 *is the structure sheaf over* M, *and* $\mathcal{M} = \mathcal{M}(M) = \Gamma(M, \mathcal{M})$ *is the sheaf over* M *constructed through coordinate chart.*

Definition 2.7. *We define the order of a meromorphic 1-form* ω *at a point* $p \in U_\alpha \subset M$ *by*

$$\nu_p(\omega) := \nu_{z_\alpha(p)}(f_\alpha).$$

Let $D(M)$ be the free abelian group with basis the points in M.

Definition 2.8. *A* ***divisor*** D *is an element of* $D(M)$:

$$D = \sum_{p \in M} v_p(D)\, p,$$

where $v_p(D) \in \mathbb{Z}$ *and almost all* $v_p(D) = 0$. *A divisor* D *is* ***effective*** *(or* ***positive****) if* $v_p(D) \geq 0$ *for all* $p \in M$. *Say two divisor* $D = \sum m_p p$ *and* $E = \sum n_p p$ *on* M *have the relation* $D \geq E$ *if* $m_p \geq n_p$ *for all* p.

Definition 2.9. *The* ***degree*** *of* D *is defined to be*

$$\deg(D) = \sum_{p \in M} v_p(D).$$

Definition 2.10. *For a nonzero meromorphic function* $f \in \mathcal{K}(M)$*, define the* ***divisor of*** f *by*

$$\mathrm{Div}(f) = \sum_{p \in M} \mathrm{ord}_p(f)\, p.$$

Divisors of the form $\mathrm{Div}(f)$ *are called* ***principal divisor****.*

Example 2.11. *The divisor* (ω) *of a holomorphic 1-form* ω *is effective.*

Definition 2.12. *For any* D *divisor of* M*, we define*

$$\mathcal{L}(D) \coloneqq \{f \in \mathcal{K}(M)^* \mid (f) + D \geq 0\} \cup \{0\},$$

and

$$\mathcal{J}(D) \coloneqq \{\omega \in \mathcal{K}^1(M)^* \mid (\omega) \geq D\} \cup \{0\},$$

where the "$*$" means excluding the zero function in $\mathcal{K}(M)$*. Also set*

$$l(D) \coloneqq \dim \mathcal{L}(D);$$
$$i(D) \coloneqq \dim \mathcal{J}(D).$$

Example 2.13. *By Liouville's Theorem, we have* $\mathcal{L}(0) \cong \mathbb{C}$*.*

The "$\cup\{0\}$" in the definition above is to make it a vector space. The next step is to define an equivalent relation on divisors.

Definition 2.14. *Say divisors* D, E *are* ***rationally equivalent*** *iff there exists* $f \in \mathcal{K}(M)^*$ *with* $(f) = D - E$*, denoted by* $D \sim E$*.*

Proposition 2.15. *If* $D \sim E$*, then*

1. $\deg(D) = \deg(E)$,
2. $\mathcal{L}(D) \cong \mathcal{L}(E)$,
3. $\mathcal{J}(D) \cong \mathcal{J}(E)$,
4. $l(D) = l(E)$,
5. $i(D) = i(E)$.

Remark 2.16. *In modern language, the Picard group* $\mathrm{Pic}(M)$ *is just*

$$\mathrm{Pic}(M) = \mathrm{Div}(M)/\sim,$$

where $\mathrm{Div}(M)$ *is the collection of divisors on* M*.*

Definition 2.17. *A* ***canonical divisor*** K *is the divisor of any meromorphic 1-form* $\omega \in \mathcal{K}^1(M)$*. Since any two such are rationally equivalent, there is a single canonical divisor class* $[K] \in \mathrm{Pic}(M)$*.*

Theorem 2.18 (Brill-Noether reciprocity)**.** *For arbitrary divisor* D*,* K *a canonical divisor, we have*

$$\mathcal{J}(D) \cong \mathcal{L}(K - D),$$

and so $i(D) = l(K - D)$*.*

Proof. Let $K - (\omega)$; if $(f) + K - D \geq 0$, then $(f\omega) = (f) + K \geq D - K + K = D$. So $f \mapsto f\omega$ maps $\mathcal{L}(K - D) \to \mathcal{J}(D)$, and $\eta \mapsto \frac{\eta}{\omega}$ gives an inverse. □

3 Statement of the proof

We are now ready to give a proof of Riemann-Roch theorem. Before we state the proof, we need some lemmas and a big theorem (we won't prove it).

Definition 3.1. *A projective algebraic curve $C \subset \mathbb{P}^2$ of degree d is the zero of a homogeneous polynomial F of degree d with 3 variables.*

Definition 3.2. *C is said to be irreducible iff F has no proper homogeneous factors.*

Definition 3.3. *ODP, or ordinary double point, is a point in the algebraic curve locally looks like the cross point in the shape "8".*

Theorem 3.4 (Normalization Theorem)**.** *We have an 1-1 correspondence between Riemann surfaces and algebraic curves in the following senses:*

1. *Given an irreducible algebraic curve $C \subset \mathbb{P}^2$, there exists a Riemann surface M and a holomorphic map $\sigma : M \to \mathbb{P}^2$ with C as its image which is 1-1 on $\sigma^{-1}(C \backslash \mathrm{sing}(C))$, where $\mathrm{sing}(C)$ is the collection of singularities on C.*

2. *Given a Riemann surface M, there exists a holomorphic map $\sigma : M \to \mathbb{P}^2$ such that*

   - *$\sigma(M)$ is an irreducible algebraic curve with $\mathrm{sing}(\sigma(M))$ consisting of ODPs (or empty).*
   - *σ is 1-1 off the preimage of these ODPs.*

So it is reasonable to associate the Riemann surfaces with algebraic curves. We will make a convention below:

Notation 3.5. *In this section, we take C to be an irreducible degree d projective algebraic curve with ODP singularities $S = \{p_1, p_2, \cdots, p_\delta\}$, and $\sigma : M := \tilde{C} \to \mathbb{P}^2$ be its normalization with $\sigma^{-1}(p_i) = \{q_i, r_i\}$. Also we define a divisor*

$$\xi := \sigma^{-1}(S) = \sum_{i=1}^{2\delta} q_i + r_i$$

of degree 2δ. Given any line $H \subset \mathbb{P}^2$, write

$$\mathcal{H} \mapsto \sigma^{-1}(H \cdot C) \in \mathrm{Div}(M)$$

for the intersection divisor.

In order to give a proof of Riemann-Roch, we still need two more lemmas, which we'll not prove here.

Lemma 3.6. *For all sufficiently large $m \in \mathbb{N}$, we have*

$$l(m\mathcal{H} - \eta) \geq md - 2\delta - g + 1$$

and

$$i(m\mathcal{H} - \eta) = 0,$$

where $g = \frac{(d-1)(d-2)}{2} - \delta$ is the genus of M.

Lemma 3.7. *Let D be a divisor on M, and $p \in M$. Then*

$$0 \leq l(D+p) - l(D) - i(D+p) + i(D) \leq 1.$$

Theorem 3.8 (Riemann-Roch Theorem)**.** *Let M be a Riemann surface of genus g, D a divisor on M, then*

$$l(D) - i(D) = \deg(D) - g + 1.$$

Note that by Brill-Noether reciprocity (Theorem 2.18), we have $i(D) = l(K - D)$. So the theorem above can be written into $l(D) - l(K-D) = \deg(D) - g + 1$, which corresponds to the form in Theorem 1.1 in the Introduction section.

Proof. By Part 2 of Normalization Theorem (Theorem 3.4, 2), we can assume we are in the situation described in the Notation 3.5, with $M = \tilde{C}$.

By Lemma 3.6, there exists $m_0 \in \mathbb{Z}$ such that for $m \geq m_0$, we have

$$l(m\mathcal{H} - \eta) - i(m\mathcal{H} - \eta) \geq md - 2\delta - g + 1.$$

Now for any two lines H_1, H_2, we have $\mathcal{H}_1 \sim \mathcal{H}_2$. So if $H_1, \cdots, H_m$ are lines in $\mathbb{P}^2$, we have by Proposition 2.15, 4 and 5 that

$$l(\mathcal{H}_1 + \cdots + \mathcal{H}_m - \eta) - i(\mathcal{H}_1 + \cdots + \mathcal{H}_m - \eta) \geq md - 2\delta - g + 1.$$

Taking m large enough and lines through all points of S and all points in D, we can ensure that $\mathcal{H}_1 + \cdots + \mathcal{H}_m - \eta - D$ is effective, so that

$$\mathcal{H}_1 + \cdots + \mathcal{H}_m - \eta = D + p_1 + \cdots + p_k,$$

where $k = md - 2\delta - \deg(D)$, and p_j's points of M. Therefore we have

$$l\left(D + \sum_{j=1}^{k} p_j\right) - i\left(D + \sum_{j=1}^{k} p_j\right) \geq k + \deg(D) - g + 1.$$

Repeatedly applying right-hand side of inequality of Lemma 3.7, we obtain

$$k + l(D) - i(D) \geq l\left(D + \sum_{j=1}^{k} p_j\right) - i\left(D + \sum_{j=1}^{k} p_j\right),$$

and we conclude that

$$l(D) - i(D) \geq \deg(D) - g + 1. \tag{1}$$

Next we show the reverse inequality. Plugging $K - D$ into (1) gives

$$l(K-D) - i(K-D) \geq \deg(K-D) - g + 1,$$

which becomes by Brill-Noether reciprocity (Theorem 2.18) that

$$i(D) - l(D) \geq 2g - 2 - \deg(D) - g + 1 = -(\deg(D) - g + 1),$$

so that

$$l(D) - i(D) \leq \deg(D) - g + 1. \tag{2}$$

Combine (1) and (2), we're done the proof! □

4 Applications

There are a lot of applications of Riemann-Roch theorem beside the one we mentioned in the Introduction section. We will briefly state two simple applications towards it.

Corollary 4.1. *We directly obtain the* ***Riemann inequality****, which is originally proved by Riemann himself in 1850's before his student Roch finally making it into equality:*

$$l(D) \geq \deg(D) - g + 1,$$

and by taking $D = 0$, we obtain

$$\dim \Omega^1(M) = g.$$

Proposition 4.2. *Up to isomorphism, there is only one Riemann surface of genus 0, which is $\mathbb{P}^1$.*

Proof. $\mathbb{P}^1$ is a Riemann surface of genus 0 by Example 2.2. Suppose M is another Riemann surface with genus 0. Then Corollary 4.1 tells us $\dim \Omega^1(M) = 0$. If we take for some $p \in M$ that $D = p$, then $\mathcal{J}(D) \subset \Omega^1(M) = \{0\}$, which gives $i(D) = 0$. By Riemann-Roch,

$$l(D) = \deg(D) - g + 1 = 1 - 0 + 1 = 2.$$

Now $\mathcal{L}(D)$ consists of functions with a (and only) simple pole allowed at p. The constant function 1 belongs to $\mathcal{L}(D)$; and since $\dim \mathcal{L}(D) = 2$, there is also a non-constant function $f \in \mathcal{L}(D)$, which by Liouville's Theorem must have the allowed simple pole. Therefore the mapping degree of $f : M \to \mathbb{P}^1$ is

$$\deg(f) = \deg(f^{-1}[\infty]) = \deg(p) = 1,$$

which implies f is an isomorphism. □