The Number $N(T)$ and $N(T, \chi)$

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1 The Number $N(T)$

Let $N(T)$ denote the number of zeros of $\zeta(s)$ in the rectangle $0 < \sigma < 1$, $0 < t \leq T$. In this section, we will prove the following approximate formula for $N(T)$:

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1}$$

Our primary tool for this is Argument principle. We will state as follow:

Theorem 1 (**Argument Principle**)**.** Suppose that $f(z)$ be a meromorphic function defined inside and on a simple closed contour C with no zeroes or poles C. Let N and P be the number of zeroes and poles, respectively, of $f(z)$ inside C, where a multiple zero or pole is counted according to its multiplicity. Then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \mathrm{d}z = N - P = \frac{1}{2\pi} \Delta_C \arg f(z).$$

We omit the proof here. It is convenient to work initially with $\xi(s)$ rather than with $\zeta(s)$ because of its simple functional equation, namely $\xi(1-s) = \xi(s)$. Assuming for simplicity that T (which we suppose to be large) does not coincide with the ordinate of a zero, we have by argument principle

$$2\pi N(T) = \Delta_R \arg \xi(s),$$

where R is the rectangle in the s-plane with vertices at 2, $2 + iT$, $-1 + iT$, -1 described in the positive sense, and in which $\xi(s)$ has no poles.

By simple observation, there is no change in $\arg \xi(s)$ as s moves along the bottom edge of rectangle since $\xi(s)$ is real and nowhere 0. Further, the change as s moves from $\frac{1}{2} + iT$ to $-1 + iT$ and then to -1 is equal to the change as s moves from 2 to $2 + iT$ and then to $\frac{1}{2} + iT$, since

$$\xi(\sigma + it) = \xi(1 - \sigma - it) = \overline{\xi(1 - \sigma + it)}.$$

Hence

$$\pi N(T) = \Delta_L \arg \xi(s), \tag{2}$$

where L denotes the line from 2 to $2 + iT$ and then to $\frac{1}{2} + iT$.

Recall that

$$\xi(s) = (s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s+1)\zeta(s).$$

Thus in (2) we have

$$\arg \xi(s) = \arg(s-1) + \arg \pi^{-\frac{1}{2}s} + \arg \Gamma(\frac{1}{2}s+1) + \arg \zeta(s). \tag{3}$$

Simple calculation gives

$$\begin{aligned}
&\Delta_L \arg(s-1) = \Delta_L \arg(\sigma - 1 + it) = \Delta_L \arctan \frac{t}{\sigma - 1} = \frac{\pi}{2} + O(T^{-1}), \\
&\Delta_L \arg \pi^{-\frac{1}{2}s} = \Delta_L(-\frac{1}{2}t \log \pi) = -\frac{1}{2}T \log \pi.
\end{aligned}$$

Recall that the Stirling's formula gives:

Theorem 2 (Stirling's formula).

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}) \tag{4}$$

Apply this to (3) we have

$$\begin{aligned}
\Delta_L \arg \Gamma(\frac{1}{2}s+1) =& \arg \Gamma(\frac{1}{2}iT + \frac{5}{4}) - \arg \Gamma(2) \\
=& \arg \left[\exp \left(\Re \log \Gamma(\frac{1}{2}iT + \frac{5}{4}) + i \Im \log \Gamma(\frac{1}{2}iT + \frac{5}{4}) \right) \right] \\
=& \Im \log \Gamma(\frac{1}{2}iT + \frac{5}{4}) \\
=& \Im \left[(\frac{1}{2}iT + \frac{3}{4}) \log(\frac{1}{2}iT + \frac{3}{4}) - \frac{1}{2}iT - \frac{5}{4} + \frac{1}{2}\log 2\pi + O(T^{-1}) \right] \\
=& \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} + \frac{3\pi}{8} + O(1).
\end{aligned}$$

Hence

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(1), \tag{5}$$

where

$$\pi S(T) = \Delta_L \arg \zeta(s) = \arg \zeta(\frac{1}{2} + iT).$$

Suffice to prove

$$\arg \zeta(\frac{1}{2} + iT) = O(\log T). \tag{6}$$

In order to prove that, we need a lemma first.

Lemma 1. If $\rho = \beta + i\gamma$ runs through the nontrivial zeros of $\zeta(s)$, then for large T

$$\sum_{\rho} \frac{1}{1 + (T-\gamma)^2} = O(\log T). \tag{7}$$

Proof. We have proved during discussion about zero-free region for $\zeta(s)$ that

$$-\Re\frac{\zeta'(s)}{\zeta(s)} < A\log t - \sum_\rho \Re\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{8}$$

for $1 \le \sigma \le 2$ and $t \ge 2$, with the sum over ρ is positive. Take $s = 2+iT$. Also we have

$$-\frac{\zeta'(s)}{\zeta(s)} \le \frac{1}{\sigma-1} + A_1,$$

where A_1 is some absolute constant. Hence $\left|\frac{\zeta'(s)}{\zeta(s)}\right|$ is bounded, and we obtain

$$\sum_\rho \Re\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) < A_2\log T,$$

Since

$$\Re\frac{1}{s-\rho} = \frac{2-\beta}{(2-\beta)^2+(T-\gamma)^2} \ge \frac{1}{4+(T-\gamma)^2},$$

we obtain the assertion in the lemma. □

Corollary 1. The number of zeros with $T-1<\gamma<T+1$ is $O(\log T)$.

Corollary 2. The sum $\sum (T-\gamma)^{-2}$ extended over the zeros with γ outside the interval just mentioned is also $O(\log T)$.

Lemma 2. For large t which does not coincide with the ordinate of a zero and $-1 \le \sigma \le 2$,

$$\frac{\zeta'(s)}{\zeta(s)} = {\sum_\rho}' \frac{1}{s-\rho} + O(\log t), \tag{9}$$

where the sum is limited to those ρ for which $|t-\gamma|<1$.

Proof. Recall that we have

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'(\frac{1}{2}s+1)}{\Gamma(\frac{1}{2}s+1)} + \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right).$$

Apply it at s and at $2+it$ and then subtracted,

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log t) + \sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{2+it-\rho}\right).$$

Note for the terms with $|\gamma-t|\ge 1$, we have

$$\left|\frac{1}{s-\rho}+\frac{1}{2+it-\rho}\right| = \frac{2-\sigma}{|(s-\rho)(2+it-\rho)|} \le \frac{3}{|\gamma-t|^2},$$

and the sum of these is $O(\log t)$ by **Corollary 2** above. As for the terms with $|\gamma-t|<1$, we have $|2+it-\rho|\ge 1$, and the number of terms is $O(\log t)$ by **Corollary 1** above. Hence the result. □

The desired estimate (6) follows from (9). To be explicit,

$$
\begin{aligned}
\pi S(T) =& \arg\zeta(\frac{1}{2}+iT) = \Im\left[\log\zeta(\frac{1}{2}+iT)\right] \\
=& \left(\int_{2}^{2+iT} - \int_{\frac{1}{2}+iT}^{2+iT}\right)\Im\left[\frac{\zeta'(s)}{\zeta(s)}\right]\mathrm{d}s \\
=& O(1) - \int_{\frac{1}{2}+iT}^{2+iT}\Im\left[\frac{\zeta'(s)}{\zeta(s)}\right]\mathrm{d}s
\end{aligned}
$$

Now from (9) we have

$$
\int_{\frac{1}{2}+iT}^{2+iT}\Im\left[\frac{1}{s-\rho}\right]\mathrm{d}s = \Delta\arg(s-\rho),
$$

and this absolute value at most π. The number of terms in the sum in (9) is $O(\log T)$, and therefore (6) follows.

Corollary 3. If the ordinates $\gamma > 0$ are enumerated in increasing order as $\gamma_1, \gamma_2, \cdots$, then $\gamma_n \sim \frac{2\pi n}{\log n}$ as $n \to \infty$.

Note. It does not follow that $\gamma_{n+1} - \gamma_n \to 0$, proved by Littlewood in 1924.

2 The Number $N(T,\chi)$

Let χ be a primitive character to the modulus q, and let $N(T,\chi)$ denote the number of $L(s,\chi)$ in the rectangle $0 < \sigma < 1$, $|t| < T$, where $T \geq 2$. Our aim in this section is to prove the following approximate formula for $N(T,\chi)$:

$$
\frac{1}{2}N(T,\chi) = \frac{T}{2\pi}\log\frac{qT}{2\pi} - \frac{T}{2\pi} + O(\log T + \log q). \tag{10}
$$

Note. It is not appropriate to consider only the upper half-plane since the zeros are not in general symmetrically placed w.r.t. the real axis. The factor $\frac{1}{2}$ serves only for the purpose of comparison with $N(T)$. Same as before, we here consider the function $\xi(s,\chi)$ instead of $L(s,\chi)$.

The proof is almost the same, but it is now convenient to consider the variation in $\arg\xi(s,\chi)$ as s describes the rectangle R with vertices at $\frac{5}{2} - iT$, $\frac{5}{2} + iT$, $-\frac{3}{2} - iT$, $-\frac{3}{2} + iT$, so as to avoid the possible zero at $s = -1$. This rectangle includes just one trivial zero of $L(s,\chi)$, at either $s = 0$ or $s = -1$, and therefore

$$
2\pi\left[N(T,\chi) + 1\right] = \Delta_R \arg(s,\chi).
$$

Recall that

$$
\xi(1-s,\overline{\chi}) = \frac{i^{\alpha}q^{\frac{1}{2}}}{\tau(\chi)}\xi(s,\chi), \tag{11}
$$

and

$$\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{s+\alpha}{2}} \Gamma(\frac{s+\alpha}{2}) L(s,\chi), \tag{12}$$

where α is 0 when $\chi(-1) = 1$ and 1 when $\chi(-1) = -1$. Then contribution of the left half of the contour is equal to that of the right, since

$$\arg \xi(\sigma + it, \chi) = \arg \overline{\xi(1 - \sigma + it, \chi)} + c,$$

where c is independent of s. Similarly, simple calculation gives (here we also use the Stirling's formula (4))

$$\Delta_L \arg \left(\frac{q}{\pi}\right)^{\frac{s+\alpha}{2}} = T \log \frac{q}{\pi},$$
$$\Delta_L \arg \Gamma(\frac{s+\alpha}{2}) = T \log \frac{T}{2} - T + O(1),$$

where L denotes the half of the contour R. This implies

$$\pi \left[N(T,\chi) + 1\right] = T \log \frac{qT}{2\pi} - T + S(T,\chi) + O(1), \tag{13}$$

where

$$\pi S(T,\chi) = \Delta_L \arg L(s,\chi).$$

Suffice to prove

$$S(T,\chi) = O(\log T + \log q). \tag{14}$$

Same as before, we need two lemmas with small modifications. Let's first state them.

Lemma 3. If $\rho = \beta + i\gamma$ runs through the nontrivial zeros of $L(s,\chi)$, where χ is primitive, then for any real t,

$$\sum_\rho \frac{1}{1 + (t-\gamma)^2} = O(\log q(|t| + 2)). \tag{15}$$

Lemma 4. For t which does not coincide with the ordinate of a zero, and $-1 \leq \sigma \leq 2$,

$$\frac{L'(s,\chi)}{L(s,\chi)} = {\sum_\rho}' \frac{1}{s - \rho} + O(\log q(|t| + 2)), \tag{16}$$

where the sum is limited to those ρ for which $|t - \gamma| < 1$.

They are proved in the same method as before, under the inequality proved during the discussion about zero-free region for $L(s,\chi)$ that

$$-\Re \frac{L'(s,\chi)}{L(s,\chi)} < c \log(q(|t| + 2)) - \sum_\rho \Re \frac{1}{s - \rho}. \tag{17}$$

Again use the same method before, we can obtain (14) from (17).

At last, if χ is not primitive, induced by the primitive character $\chi_1(\mod q_1)$, then (10) remains valid for $N(T,\chi)$ as defined, provided we replace q by q_1. But if $N_R(T,\chi)$ denotes the number of zeros in the rectangle R defined before, we must include the zeros on $\sigma = 0$ of

$$\prod_{p|q} \left[1 - \chi_1(p)p^{-s}\right],$$

according to

$$L(s,\chi) = \prod_{p\nmid q} \left[1 - \chi(p)p^{-s}\right]^{-1} = L(s,\chi_1) \prod_{p|q} \left[1 - \chi_1(p)p^{-s}\right]. \tag{18}$$

These are (for each p not dividing q_1) spaced at equal distances $\frac{2\pi}{\log p}$ apart. Their number, with $|t| < T$, is

$$O\left[\sum_{p|q}(T\log p + 1)\right] = O(T\log q).$$

Hence

$$N_R(T,\chi) = \frac{T}{\pi}\log\frac{T}{2\pi} + O(T\log q), \tag{19}$$

for $T \geq 2$.