Trace Method

Outline

- Edgewise subdivision, definition of THH via FSP.
- TC, definition of cyclotomic trace.
- Computation of TC. Info from (p-) cyclotomic spectra.

I. THH, as cyclic space.

1. Cyclic category, objects.

Δ cat w/ $[n] = \{0, 1, \dots, n\}$ morphism = order-preserving map.

C_n = cyclic gp of order n. Consider the new cat $\Delta\mathcal{C}$, w/

- obj = obj. of Δ.
- mor = d_i, s_j faces / degeneracies. &

  cyclic operators $\tau_n : [n] \longrightarrow [n]$ s.t. $\tau_n^{n+1} = id$.

$$(*) \quad \begin{cases} \tau_n d_i = d_{i-1} \tau_{n-1}, & 1 \le i \le n \\ \tau_n s_i = s_{i-1} \tau_{n+1}, & 1 \le i \le n \end{cases}$$

Prop. $B\Delta\mathcal{C} = |\Delta\mathcal{C}| = K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$.

Prop. 1) $\mathrm{Hom}_{\Delta\mathcal{C}}([n], [n]) = C_{n+1}$

2) $\forall f: [n] \longrightarrow [m]$ in $\Delta\mathcal{C}$. $\exists! \ \phi \circ g$ s.t. $f = \phi \circ g$.

$\phi \in \mathrm{Hom}_\Delta([n], [m])$ and $g \in \mathrm{Aut}_{\Delta\mathcal{C}}([n]) = \mathbb{Z}/_{n+1}$.

Consider $X : \Delta\mathcal{C}^{op} \longrightarrow$ Set / Space (CGWH), this is a cyclic object.

$(*) \Rightarrow \tau_n s_0 = s_n \tau_{n+1}^2 \Rightarrow \tau_n^{n+1} s_i = s_i \tau_{n+1}^{n+2}$. Similarly, $\tau_n^{n+1} d_i = d_i \tau_{n-1}^{n}$

- Λ_r = cat s.t. $\Delta\mathcal{C} \subset \Lambda_r$, and $\tau_n^{r(n+1)} = id$.

  obj = cyclic object if $r = 1$ by def.

standard simplex is $\Lambda^n_r := \mathrm{Hom}_{\Lambda_r}(-, [n])$.

Prop. $|\Lambda^n_r| \cong \mathbb{R}/r\mathbb{Z} \times \Delta^n$, when $r=1$, it's S^1-action.

Cor. $\forall$ Λ_r-obj X has a canonical action of $\mathbb{R}/r\mathbb{Z}$, hence a S^1-action identifying $\theta + r\mathbb{Z}$ w/ $e^{2\pi i\theta/r}$. One can check the geometric realization agrees:

$$\begin{aligned} |X| &= \coprod X_n \times \Delta^n/\sim \\ &\wr\wr \\ |X|_{\Lambda_r} &= \coprod X_n \times \Lambda^n_r/\sim \end{aligned}$$

2. Edgewise subdivision

Let $X \in sSet$. $r \in \mathbb{N}$.

Def. edgewise subdivision $sd_r : \Delta \longrightarrow \Delta$

$$[m-1] \longmapsto [mr-1]$$

$$f \longmapsto \coprod^r f \quad \text{w/}$$

$$sd_r(f)(am+b) = an + f(b).$$

for $0 \le a < r$, $0 \le b < m$,

$f: [m-1] \longrightarrow [n-1]$

The subdivision of X is $sd_r X = X \circ sd_r$. s.t.

$$sd_r X_n = X_{(n+1)r-1}.$$

w/ $\bar{d_i} : sd_r X_n \longrightarrow sd_r X_{n-1}$

$\bar{s_i} : sd_r X_n \longrightarrow sd_r X_{n+1}$

and $\bar{d_i} = d_i \circ d_{i+(n+1)} \circ \ldots \circ d_{i+(r-1)(n+1)}$

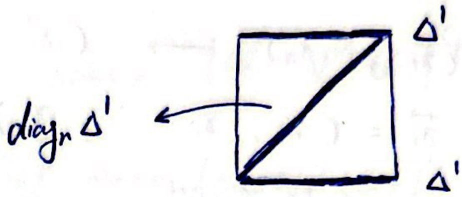
$\bar{s_i} = s_{i+(r-1)(n+2)} \circ \cdots \circ s_{i+(n+2)} \circ s_i$

- The standard simplex Δ^{rm-1} = j-fold join of Δ^{m-1} w/ itself.

Note $[n] * [m] = [n+m+1]$.

diagonal emb: $\mathrm{diag}_r : \Delta^{m-1} \longrightarrow \Delta^{rm-1}$

$$u \longmapsto \tfrac{1}{r}u \oplus \tfrac{1}{r}u \oplus \cdots \oplus \tfrac{1}{r}u.$$



Rk. This diagonal gives a map from X to $\mathrm{sd}_r X$.

- $|\mathrm{sd}_r X| \simeq |X|$ (Look at diagonal)

3. Cyclic bar construction.

G top gp, mostly assumed to be cpt (e.g. Lie gps, finite gps). $X \in G\mathrm{Top}$.

Def. Cyclic bar construction of X rel G is

$$N_n^{cyc}(X;G) = X \times G^n$$

w/ faces & degeneracies

$$d_0(x, g_1, \dots, g_n) = (xg_1, g_2, \dots, g_n).$$

$$d_1(x, g_1, \dots, g_n) = (g_n x, g_1, \dots, g_{n-1})$$

$$d_i(x, g_1, \dots, g_n) = (x, g_1, \dots, g_i g_{i+1}, \dots, g_n)$$

$$s_i(x, g_1, \dots, g_n) = (x, g_1, \dots, g_i, 1, g_{i+1}, \dots, g_n).$$

Set $t_n(g_1, \dots, g_n) = (g_n, g_1, \dots, g_{n-1})$. Under t_n, $N_n^{cyc}(X;G)$ is a cyclic space.

- Interaction w/ sd_r.

$\mathrm{sd}_r N_\bullet^{cyc}(X) \cong N_\bullet^{cyc}(t(X^r); G^r)$, where $t(X^r) = X^r$ w/ a twisted two-sided G^r-str:

$$(x_1, \dots, x_r) \cdot (g_1, \dots, g_r) = (x_1 g_1, \dots, x_r g_r)$$

$$(g_1, \dots, g_r) \cdot (x_1, \dots, x_r) = (g_r x_1, g_1 x_2, \dots, g_{r-1} x_r)$$

This is also a cyclic bar construction.

Consider the associated diagonal embedding :

$$\Delta_{r,\bullet} : N^{cyc}_{\bullet}(X) \longrightarrow sd_r N^{cyc}_{\bullet}(X)$$

$$(x_1, \dots, x_n) \longmapsto (\vec{x_1}, \dots, \vec{x_n})$$

where $\vec{x_i} = (x_i, x_i, \dots, x_i) \in X^r$.

Now C_r-action, gen. by $t^{n+1}_{(n+1)r-1}$ on n-simplices, corresponds to permutation action on X^r. Thus taking the fixed point,

$$\Delta_{r,\bullet} : N^{cyc}_{\bullet}(X) \longrightarrow \left(sd_r N^{cyc}_{\bullet}(X)\right)^{C_r}$$ gives a simplicial iso.

Prop$^{(*)}$. $\exists$ htpy comm. diagram for any top gp G :

$$\begin{tikzcd}
BG = {|N_\bullet(G)|} \arrow[r, "i"] \arrow[d, "i"'] & {|N^{cyc}_\bullet(G)|} \arrow[r, "\Delta_{rs,\bullet}"] & {|sd_{rs} N^{cyc}_\bullet(G)|^{C_{rs}}} \arrow[d, hook, "\iota = \text{inclusion}"] \\
{|N^{cyc}_\bullet(G)|} \arrow[r, "\Delta_{s,\bullet}"] & {|sd_s N^{cyc}_\bullet(G)|^{C_s}} & {|sd_{rs} N^{cyc}_\bullet(G)|^{C_s}} \arrow[l, "D_r"']
\end{tikzcd}$$

where $i : \underbrace{N_\bullet(G)}_{\substack{\text{bar construction} \\ \text{(i.e. w/ no } t_n)}} \longrightarrow N^{cyc}_{\bullet+1}(G)$, $(g_1, \dots, g_n) \longmapsto ((\prod g_i)^{-1}, g_1, \dots, g_n)$

D_r is a htpy equiv. See [BMH, Chapter 2].

- By def. $\exists$ comm. diagram up to htpy :

$$\begin{tikzcd}
{|sd_{p^n} N^{cyc}_\bullet(G)|^{C_{p^n}}} \arrow[r, "\Delta_{p,\bullet}"] \arrow[d, "D"'] & {|sd_{p^{n+1}} N^{cyc}_\bullet(G)|^{C_{p^{n+1}}}} \arrow[d, "D"] \\
{|sd_{p^{n-1}} N^{cyc}_\bullet(G)|^{C_{p^{n-1}}}} \arrow[r, "\Delta_{p,\bullet}"] & {|sd_{p^n} N^{cyc}_\bullet(G)|^{C_{p^n}}}
\end{tikzcd} \qquad (\bigstar\bigstar)$$

where $D : |sd_{p^n} N^{cyc}_\bullet(G)|^{C_{p^n}} \xhookrightarrow{\iota} |sd_{p^n} N^{cyc}_\bullet(G)|^{C_{p^{n-1}}} \xrightarrow{D_p} |sd_{p^{n-1}} N^{cyc}_\bullet(G)|^{C_{p^{n-1}}}$

By Prop$^{(*)}$, we have a projective system $(|sd_{p^n} N_\bullet^{cyc}(G)|^{C_p}, D)$, and hence a map

$$I: |N_\bullet G| \longrightarrow \underleftarrow{holim}_D |sd_{p^n} N_\bullet^{cyc}(G)|^{C_{p^n}}$$

Note that $\Delta_{p,\bullet}: N_\bullet^{cyc}(G) \longrightarrow (sd_p N_\bullet^{cyc}(G))^{C_p}$ is a simplicial iso

It induces an homeo after passing to geometric realization. Abuse the notation, we have $\Delta_{p,\bullet}: |sd_{p^n} N_\bullet^{cyc}(G)|^{C_{p^n}} \xrightarrow{\cong} |sd_{p^{n+1}} N_\bullet^{cyc}(G)|^{C_{p^{n+1}}}$.

Write $\Phi_p = \Delta_{p,\bullet}^{-1}$, we have a map (a htpy class)

$$I: |N_\bullet G| \longrightarrow \left(\underleftarrow{holim}_D |sd_{p^n} N_\bullet^{cyc}(G)|^{C_{p^n}}\right)^{h\Phi_p},$$

where $h\Phi_p$ = htpy equalizer of Φ_p and id. (Indeed the htpy fixed pts)

Rk If G is only a monoid, then applying Grothendieck completion $G \longrightarrow \hat{G}$, then do the same thing.

4. THH of FSP.

Def FSP, or functor with smash product, is a functor $F: Top_* \longrightarrow Top_*$ w/ 1) $1_X: X \longrightarrow F(X)$

2) $\mu_{X,Y}: F(X) \wedge F(Y) \longrightarrow F(X \wedge Y)$ s.t.

① $\mu_{X,Y}(1_X \wedge 1_Y) = 1_{X \wedge Y}$

② $\mu_{X\wedge Y, Z}(\mu_{X,Y} \wedge id_{F(Z)}) = \mu_{X, Y\wedge Z}(id_{F(X)} \wedge \mu_{Y,Z})$

③ $F(T) \circ \mu_{X,Y} \circ 1_X \wedge id_{F(Y)} = \mu_{Y,X} \circ (id_{F(Y)} \wedge 1_X) \circ T$

where $T: X \wedge Y \longrightarrow Y \wedge X$.

So ② = associativity ③ = commutativity.

Def THH functor of FSP F is

$$THH_\bullet(F) = \left([p] \mapsto \underrightarrow{\mathrm{hocolim}}_{(\Delta)^p} Map(S^{i_0} \wedge \cdots \wedge S^{i_p}, F(S^{i_0}) \wedge \cdots \wedge F(S^{i_p})) \right)$$

This is a simplicial space, $i_0, \dots, i_p \in \mathbb{N}$.

Rk 1) $\pi_r \Omega^i F(S^i X) \longrightarrow \pi_r \Omega^{i+1} F(S^{i+1} X)$, S = unduced suspension.

2) $THH(F) = | THH_\bullet(F) |$ is the topological Hochschild space.

- Edgewise subdivision of THH

Let R = (complex) regular representation of C_r. $iR = \underbrace{R \oplus \cdots \oplus R}_{i}$. Let

$$sd_r THH_\bullet(F) = \left([p] \mapsto \underrightarrow{\mathrm{hocolim}}_{(\Delta)^p} Map(S^{i_0R} \wedge \cdots \wedge S^{i_pR}, F(S^{i_0})^{(r)} \wedge \cdots \wedge F(S^{i_p})^{(r)}) \right).$$

where S^{i_jR} = one pt compactification of i_jR. $j = 0, 1, \dots, p$.

$= (S^{i_j})^{\wedge r}$ as C_r-space.

This is b/c over $\mathbb{C}$. $R = \left(\begin{array}{l} \rho : C_r \longrightarrow GL(V) \\ g \longmapsto \rho(g) : v \longmapsto e^{2\pi i k/r} v, \ k = 1, \dots, n \end{array} \right).$

hence $\forall$ elet of C_r gives rise to $\rho(g)$, $g \in C_r$ w/ $\rho(g)$ becomes S^1 after one-pt compactification.

Also $(-)^{(r)}$ = r-fold smash

Prp 1) $sd_r THH_\bullet(F)$ satisfies Morita equivalence, i.e. C_r-equivariant h.e.

$$|sd_r THH_\bullet(F)| \simeq_{C_r} |sd_r THH_\bullet(M_n(F))|$$

where $M_n(F)$ is the matrix functor $Map([n], [n] \wedge F(-))$.

2) $THH(F)$ is the 0^{th} part of an Ω-spectrum $tHH(F)$.

3) $\forall F_1, F_2$ FSP, then $\exists$ C_r-equivariant htpy equiv.

$$|sd_r THH_\bullet(F_1 \times F_2)| \simeq_{C_r} |sd_r THH_\bullet(F_1)| \times |sd_r THH_\bullet(F_2)|$$

4) $|\mathrm{sd}_r\, THH_\bullet(F)|$ is an equivariant Γ-space [BMH, Chaper 4].

II. Cyclotomic trace and TC

1. Dennis trace

Consider the map

$$|N_\bullet(GL_n(R))| \xrightarrow{i} |N_\bullet^{cyc}(GL_n R)| \xrightarrow{s} |N_{\otimes,\bullet}^{cyc}(M_n R)| \simeq |N_{\otimes,\bullet}^{cyc}(R)|$$

Morita equiv.

Here i is as in Prop(*), s is induced by the embedding $(GL_n R)^k \subset (M_n R)^{\otimes k}$, and $N_{\otimes,\bullet}^{cyc}(M_n R) = (M_n R) \otimes \cdots \otimes (M_n R)$ w/ faces, degeneracies & t_n. Now taking stabilization

$$GL_n \hookrightarrow GL_{n+1} \hookrightarrow \cdots$$

$$M_n \hookrightarrow M_{n+1} \hookrightarrow \cdots$$

we get pretr: $BGL(R) \longrightarrow |N_{\otimes,\bullet}^{cyc}(R)| = HH(R)$. Factoring through "+" construction of LHS, we get the Dennis trace

$$Tr: BGL(R)^+ \longrightarrow HH(R).$$

Passing to π_*:

$$tr: K_n(R) \longrightarrow HH_n(R).$$

- We can get a topological version of this map. Explicitly, consider the simplicial map $S_\bullet: N_\bullet^{cyc}(GL_n F) \longrightarrow THH_\bullet(M_n F)$

$$(f_0, \dots, f_n) \longmapsto f_0 \wedge \cdots \wedge f_n$$

where $f_i: S^{n_i} \longrightarrow M_n F(S^{n_i})$, F FSP.

By Quillen's "+"-construction, $K(F) \times \mathbb{Z} = BGL_\infty(F)^+ \times \mathbb{Z}$, $BGL_\infty(F) = \underline{\mathrm{holim}}\, BGL_k(F)$. As in classical Barratt - Priddy - Quillen, there is an ∞-loop space str. on $K(F)$. (Omitted). See [BHM, Definitio 5.4].

By Morita equiv, taking $|-|$, we have

$$S := |S_\cdot| : K^{cyc}(F) \longrightarrow THH(F) \times \mathbb{Z}.$$

and S is an C_n-equivariant map of ∞-loop spaces.

2. Cyclotomic trace and TC.

Upshot cycl. trace is an variant of Dennis trace with edgewise subdivision.

Consider the diagram derived from $(\star\star)$

$(\star\star\star)$

$$\begin{tikzcd}
\left|sd_{p^{n-1}} N^{cyc}_\bullet(GL_k F)\right|^{C_{p^{n-1}}} \arrow[r, "S"] \arrow[d, "\Delta_{p,\bullet}"] & \left|sd_{p^{n-1}} THH_\bullet(M_k F)\right|^{C_{p^{n-1}}} \\
\left|sd_{p^n} N^{cyc}_\bullet(GL_k F)\right|^{C_{p^n}} \arrow[r, "S"] & \left|sd_{p^n} THH_\bullet(M_k F)\right|^{C_{p^n}} \arrow[u, "\Phi_p = \Delta_{p,\bullet}^{-1}"']
\end{tikzcd}$$

Let R = regular rep. of C_{p^n}

$\bar{R}$ = " " " $C_{p^{n-1}}$

Since $C_{p^n}/C_p \cong C_{p^{n-1}}$, $\bar{R} \cong R^{C_p}$. Consider the map (after Morita equ)

$$\begin{tikzcd}
\mathrm{Fix}_p : \mathrm{Map}_{C_{p^n}}\left(S^{i_0 R} \wedge \cdots \wedge S^{i_k R},\ F(S^{i_0})^{(p^n)} \wedge \cdots \wedge F(S^{i_k})^{(p^n)}\right) \arrow[d] \\
\mathrm{Map}_{C_{p^{n-1}}}\left(S^{i_0 \bar{R}} \wedge \cdots \wedge S^{i_k \bar{R}},\ F(S^{i_0})^{(p^{n-1})} \wedge \cdots \wedge F(S^{i_k})^{(p^{n-1})}\right)
\end{tikzcd}$$

taking each map f to the induced map f^{C_p} on C_p-fixed points.

Fix_p induces $\Phi_{p,\bullet} : \mathrm{sd}_{p^n} \mathrm{THH}_\bullet(F)^{C_{p^n}} \longrightarrow \mathrm{sd}_{p^{n-1}} \mathrm{THH}_\bullet(F)^{C_{p^{n-1}}}$.

One can check $|\Phi_{p,\bullet}| = \Phi_p$ in $(\overset{\star}{\star\star})$.

<u>Prop</u> Back to diagram $(\star\star)$.

$$D : |\mathrm{sd}_{p^n} N_\bullet^{cyc}(GL_k F)|^{C_{p^n}} \xhookrightarrow{\iota} |\mathrm{sd}_{p^n} N_\bullet^{cyc}(GL_k F)|^{C_{p^{n-1}}} \xrightarrow{D_p} |\mathrm{sd}_{p^{n-1}} N_\bullet^{cyc}(GL_k F)|^{C_{p^{n-1}}}$$

one can check that, as $\Phi_p = \Delta_{p,\bullet}^{-1}$, one has $D\Phi_p = \Phi_p D$

<u>Def</u>. Let F be FSP, define the topological cyclic homology at p to be

$$\begin{aligned} TC(F;p) &= \left(\mathrm{holim}_D \ THH(F)^{C_{p^n}}\right)^{h\Phi_p} \\ &= \left(\mathrm{holim}_{\Phi_p} \ THH(F)^{C_{p^n}}\right)^{hD} \end{aligned}$$

here $h\Phi_p$. hD are htpy equalizers of Φ_p (resp. D) and id.

- The cyclotomic trace is given by

$$Trc : K(F) \longrightarrow TC(F;p)$$

$$Trc = \mathrm{proj} \circ S \circ I, \text{ where}$$

$$I : K(F) = |N_\bullet(GL_n F)| \longrightarrow \left(\mathrm{holim}_D |\mathrm{sd}_{p^n} N_\bullet^{cyc}(GL_n F)|^{C_{p^n}}\right)^{h\Phi_p}$$

$$S : \left(\mathrm{holim}_D |\mathrm{sd}_{p^n} N_\bullet^{cyc}(GL_n F)|^{C_{p^n}}\right)^{h\Phi_p} \longrightarrow \left(\mathrm{holim}_D THH(F)^{C_{p^n}}\right)^{h\Phi_p}$$

proj : forget the possible "$\times \mathbb{Z}$" factor.

Note S is induced by $\Phi_{p,\bullet}$ and $(\overset{\star}{\star\star})$

<u>Prop</u> TC satisfies Morita equivalence.

<u>Note</u> Taking the profinite completion, $TC(F) := TC(F;p)_p^\wedge$

<u>Thm</u> $THH(F)$. as 0^{th} part of of an Ω-spectrum $tHH(F)$.

This $tHH(F)$ is a p-cyclotomic spectrum.

III. Computation of TC, Equivariant Homotopy Theory.

1. Basics about equivariant htpy thy

Let G = cpt Lie gp. $\mathcal{U}$ = complete universe of G

= direct sum of countably many copies of irred. rep. of G.

Thm (Peter-Weyl) $\forall$ irred. rep. V of G is finite dimensional.

Endow $\mathcal{U}$ w/ inner product structure.

Def (Lewis-May coordinate-free G-spectra)

A G-prespectrum indexed in $\mathcal{U}$, denoted E, is the following data:

1) $\forall\, V \subset \mathcal{U}$, a G-space $E(V)$.

2) $\forall\, V \subset W \subset \mathcal{U}$, a structure map $\sigma_{V,W} : S^{W-V} \wedge E(V) \longrightarrow E(W)$ which is G-equivariant. $W-V$ = orthogonal completion of V in W. S^V = one pt completion of V, regarded as indexed in $\mathcal{U}$.

subject to, $\forall\, V \subset W \subset T \subset \mathcal{U}$. G-equivariant comm. diagram

$$\begin{tikzcd}
S^{T-W} \wedge S^{W-V} \wedge E(V) \arrow[r, "1 \wedge \sigma_{V,W}"] \arrow[d] & S^{T-W} \wedge E(W) \arrow[d, "\sigma_{W,T}"] \\
S^{T-V} \wedge E(V) \arrow[r, "\sigma_{V,T}"] & E(T)
\end{tikzcd}$$

E is G-spectrum if E is G-prespectrum w/

$$\tilde{\sigma}_{V,W} : E(V) \longrightarrow \Omega^{W-V} E(W) \quad \text{a homeo.}$$

- Change-of-universe functors

$f : \mathcal{U} \to \mathcal{U}'$ G-isometric embedding. It induces an adjunction:

$$f_* : GS\mathcal{U} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} GS\mathcal{U}' : f^*$$

where $(f_* E)(V) = E(f(V))$. $V \subset \mathcal{U}$.

$(f^* E)(V') = E(f^{-1}(V' \cap f(\mathcal{U}))) \wedge S^{V' - f(V)}$ + spectrification, $V' \subset \mathcal{U}'$

Write $i : \mathcal{U}^G \to \mathcal{U}$. $\mathcal{U}^G$ = G-fixed subuniverse of $\mathcal{U}$.

$GS\mathcal{U}^G$ = naive G-spectra. $GS\mathcal{U}$ = genuine G-spectra.

Change-of-universe gives an relation between $GS\mathcal{U}$ & $GS\mathcal{U}^G$.

Rk. 1) $\Sigma^\infty_G : GTop \to GS\mathcal{U}$, G-suspension spectra functor

2) Let $GP\mathcal{U}$ = cat of G-prespectra. Then Forget $= F : GS\mathcal{U} \to GP\mathcal{U}$ has a left adjoint, called spectrification, denoted $\mathcal{L}$.

$$(\mathcal{L}E)(V) = \mathrm{colim}_{V \subset W} \Omega^{W-V} E(W) \quad V \subset \mathcal{U}.$$

3) We have the function spectra, $\forall E, E' \in GS\mathcal{U}$,

$$F(E, E') \in GS\mathcal{U}.$$

Prp $X \in GTop$, $E \in GS\mathcal{U}$. then $\forall E' \in GS\mathcal{U}$,

$$\mathrm{Hom}_{GS\mathcal{U}}(E \wedge X, E') \cong \mathrm{Hom}_{GS\mathcal{U}}(E, F(E, E'))$$

Prp $GS\mathcal{U}$ is bicomplete.

Rk. If $\mathcal{U}$ consists of only trivial rep. of G, i.e. $\mathcal{U} = \bigoplus_{i=0}^{\infty} V$, or $\mathcal{U} = \mathcal{U}^G$.
then $GS\mathcal{U} = Sp$.

Def A htpy between E & F in $GS\mathcal{U}$ is a map $E \wedge I_+ \to F$.

$[E, F]_G$:= htpy class (G-equivariant) of $E \to F$.

eg. $X, Y \in GTop$. X compact, then $[\Sigma^\infty_G X, \Sigma^\infty_G Y]_G = \mathrm{colim}_V [\Sigma^V X, \Sigma^V Y]_G$.

Def $E \in GS\mathcal{U}$, then H-equivariant htpy gp of E is

$$\pi_n^H(E) = [G/H_+ \wedge S^n, E]_G \quad , n \in \mathbb{Z}.$$

▶ Local coeff: $\underline{\pi}_n(E) : \mathrm{Orb}_G \longrightarrow Ab$

$$G/H \longmapsto \pi_n^H E.$$

Th Assume all G-CW-spectra.

① $f: E \to E'$ w.e. iff $f_V: E(V) \longrightarrow E'(V)$ w.e. $\forall V \subset \mathcal{U}$

iff $\pi_n^H(f)$ iso, $\forall n$, $H \subset G$.

② (Whitehead) $E \in GS\mathcal{U}$, $f: F \longrightarrow F'$ w.e. then

$$f_* : [E, F]_G \longrightarrow [E, F']_G \text{ iso.}$$

③ (Cellular Approximation). $E, F \in GS\mathcal{U}$. A subcpx of E, $f: E \to F$

$f|_A$ cellular. Then $f \simeq$ cellular map rel. A.

④ (G-CW Approximation) $E \in GS\mathcal{U}$, then $\exists$ G-CW-spectrum F and w.e.

$f: F \to E$.

Def Fixed pt spectra (or, categorical fixed pts)

① $D \in GS\mathcal{U}^G$, then $D^G(V) = (D(V))^G$

② $E \in GS\mathcal{U}$, $E^G := (i^* E)^G$, where $i^*: GS\mathcal{U} \to GS\mathcal{U}^G$ change-of universe.

- $(-)^G : GS\mathcal{U} \to Sp$. This is homotopical in L-M G-spectra, but in other models.
- Not good: Σ^∞ and $(-)^G$ not commutes.

Th (tom Dieck splitting)

G finite or cpt Lie, then $\forall X \in Top_*$,

$$(\Sigma^\infty X)^G \simeq \bigvee_{(H)\subset G} \Sigma^\infty EWH_+ \wedge_{WH} X^H,$$

where (H) = conjugate class of $H \leq G$. $WH = N_G H / H$ is the Weyl gp.

▲ Better model: geometric fixed pts. $\Phi^G : GS\mathcal{U} \to Sp$.

Let $\mathcal{F} = \{H \subseteq G :$ H subgp. $H \neq \phi$, H closed under conjugation and subgps of H is in $\mathcal{F}\}$.

Then $\exists$ pted G-space $E\mathcal{F} \in GTop$ w/ fixed pts. $\forall H \subset G$.

$$(E\mathcal{F})^H \simeq \begin{cases} *, & H\in\mathcal{F} \\ \phi, & H\notin\mathcal{F}. \end{cases}$$

e.g. $H=\{1\} \Rightarrow (E\mathcal{F})^H = EG.$

Now require $\mathcal{F}$ to be proper, i.e. consisting of all proper subgps. We can also define $\mathcal{F}[N] = \{K\subset G : N\notin K\}$ subgps not containing N.

Def $f: E\to F$ $\mathcal{F}$-equiv. if $E\wedge E\mathcal{F}_+ \to F\wedge E\mathcal{F}_+$ w.e.

- Consider the cofiber sequence:

$$E\mathcal{F}_+ \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}}$$

Def. $\Phi^G : GS\mathcal{U} \to Sp$ geometric fixed pt functor is given by

$$\Phi^G(E) = (\widetilde{E\mathcal{F}}\wedge E)^G$$

Thm $f: E\to E'$ equiv. of G-spectra iff $\Phi^H f : \Phi^H E \to \Phi^H E'$ an non-equivariantly equiv, $\forall H\subset G$.

Def Htpy fixed pts for $E\in GS\mathcal{U}$: $E^{hG} = F(EG_+, E)^G$.

Htpy orbits : $E_{hG} = EG_+ \wedge_G E$

Thm (Wirthmüller) $E\in HS\mathcal{U}_H$. $\mathcal{U}_H$ = complete universe of $H\subset G$. There is a natural w.e. $F_H(G_+, \Sigma^{L(H)}E) \xrightarrow{\simeq} G_+\wedge_H E$

$L(H)$ = tangent H-rep. at the identity coset of G/H. i.e. rep of $G/_H$ at eH, regarded as H-rep.

Here coinduced G-spectrum : $F_H(G_+, E) = F(G/H_+, E)$

induced G-spectrum : $G_+\wedge_H E = (G/H)_+\wedge E$

e.g. $(\Sigma^\infty_G X)^{C_2} \simeq (\Sigma^\infty EC_{2+}\wedge_{C_2} X^{\{1\}}) \vee (\Sigma^\infty X^{C_2})$

$\simeq (\Sigma^\infty BC_{2+}\wedge X^{\{1\}}) \vee \Phi^{C_2}(\Sigma^\infty_{C_2} X)$

- What's good about Φ^G ?
  - $\Phi^G \circ \Sigma^\infty_G = \Sigma^\infty \circ (-)^G$
  - $\Phi^G \circ (- \wedge -) = (\Phi^G \circ -) \wedge (\Phi^G \circ -)$

Rk. Point-set model for Φ^G:

$\forall\, X \in GS\mathcal{U}$, let $\rho_G = \mathbb{R}G$ regular rep. of the (finite) gp G.

$(\Phi^G X)(V) := X(\rho_G \otimes V)^G$. $V \subset \mathcal{U}^G$.

str. map: $S^{W-V} \wedge (\Phi^G X)(V) \cong (S^{(W-V)\otimes \rho_G} \wedge X(\rho_G \otimes V))^G$

$$\longrightarrow (X(\rho_G \otimes W))^G = (\Phi^G X)(W)$$

$V \subset W$. This is actually the original version. When we use the orthogonal G-spectra for model of equivariant spectra, then this $= \Phi^G X = (\widetilde{E\mathcal{F}} \wedge X)^G$.

- Tate diagram

For $E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}$. Take $\mathcal{F} = \{1\} \Rightarrow EG_+ = E\mathcal{F}_+$. We'll use this for later. Now

$$EG_+ \longrightarrow S^0 \longrightarrow \widetilde{E\mathcal{F}}$$

Apply ① $- \wedge X$, X G-spectra.

② $- \wedge F(EG_+, X)$

③ $X \xrightarrow{\varepsilon} F(EG_+, X)$

⇝

$$\begin{tikzcd}
EG_+ \wedge X \arrow[r] \arrow[d, "id \wedge \varepsilon"] & X \arrow[r] \arrow[d, "\varepsilon"] & \widetilde{EG} \wedge X \arrow[d, "id \wedge \varepsilon"] \\
EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \widetilde{EG} \wedge F(EG_+, X)
\end{tikzcd}$$

Prp [Greenlees, May, Generalized Tate cohomology. §1.2]

$EG_+ \wedge X \simeq EG_+ \wedge F(EG_+, X)$ as G-spectra.

Then apply ④ $(-)^G$, and

<u>Prop</u> (Adams isomorphism) $(EG_+ \wedge X)^G \simeq X_{hG}$.

$\rightsquigarrow$

$$\begin{tikzcd} & X_{hG} \arrow[r] \arrow[d, "\simeq"'] & X^G \arrow[r] \arrow[d] & (\widetilde{EG} \wedge X)^G \arrow[d] & \\ (\bigstar\bigstar\bigstar\bigstar) & X_{hG} \arrow[r, "N"] & X^{hG} \arrow[r] & (\widetilde{EG} \wedge F(EG_+, X))^G & =: X^{tG} \end{tikzcd}$$

Here X^{tG} = Tate spectrum of X.

- When $G = C_{p^n}$, p prime, $\mathcal{F} = \{1\} \Rightarrow \widetilde{EG} = \widetilde{E\mathcal{F}}$, and so $(\widetilde{EG} \wedge X)^G = \Phi^G X$. We will always work in this case:

$$\begin{tikzcd} X_{hG} \arrow[r] \arrow[d, "\simeq"'] & X^G \arrow[r] \arrow[d] & \Phi^G X \arrow[d] \\ X_{hG} \arrow[r, "N"] & X^{hG} \arrow[r] & X^{tG} \end{tikzcd}$$

2. Computational tool: AHSS (Adams - Hirzebruch spectral sequence).

- $E^2_{s,t}(X_{hG}) = H_s(G; \pi_t X) \Longrightarrow \pi_{s+t}(X_{hG})$

  $E^2_{s,t}(X^{hG}) = H^{-s}(G; \pi_t X) \Longrightarrow \pi_{s+t}(X^{hG})$

  $E^2_{s,t}(X^{tG}) = \hat{H}^{-s}(G; \pi_t S) \Longrightarrow \pi_{t+s}(X^{tG})$. $\hat{H}$ = Tate cohomology.

▶ <u>Goal</u> unpack all of these spectral sequences.

2.1 Tate cohomology

<u>IDEA</u> Pack gp homology & cohomology into a single cohomology.

G finite. M G-module. The algebraic orbits & fixed pts are given by

$M_G := M / \langle g \cdot m - m : g \in G,\ m \in M \rangle \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$

$M^G := \{ m \in M : g \cdot m = m,\ \forall g \in M \} \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M)$

norm element $N_G = \sum_{g \in G} g$. $N : M_G \longrightarrow M^G$

$m \longmapsto \sum_{g \in G} g \cdot m$.

The Tate cohomology gps are

$$\hat{H}^i(G;M) = \begin{cases} H^i(G;M) & , i \geq 1 \\ H_{-i-1}(G;M) & , i \leq -2 \\ \ker N & , i = -1 \\ \operatorname{coker} N & , i = 0 \end{cases}$$

i.e. s.e.s.

$$0 \to \hat{H}^{-1}(G;M) \to H_0(G:M) \xrightarrow{N} H^0(G;M) \to \hat{H}^0(G;M) \to 0$$

e.g. $G = C_p$. $M = G$-module. It's clear that we have the free resolution

$$\cdots \to \mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

Note $M \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} M \cong \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], M)$

$$\Rightarrow H^{2n}(G;M) = H_{2n-1}(G;M) = \ker(g-1)/\operatorname{im} N_G = \operatorname{coker} N = \hat{H}^0(G;M).$$

Compute Tate cohomlgy via gp cohlgy & homlgy

2.2. AHSS, skeleton filtration.

Recall that EG contractible G-CW-cpx with free G-action. $EG_+ = EG \cup \{pt\}$.

$\widetilde{EG} = S(EG_+)$, or $S^0 \cup CEG_+$ as in cofiber seq.

Take skeleton filtration of EG_+:

$$\begin{tikzcd} * \arrow[r] & E_0G_+ \arrow[r] \arrow[d] & E_1G_+ \arrow[r] \arrow[d] & E_2G_+ \arrow[r] \arrow[d] & \cdots \\ & E_0G_+/_* & E_1G_+/E_0G_+ & E_2G_+/E_1G_+ & \end{tikzcd}$$

s.t. $\exists$ cofiber seq. $E_iG_+/E_{i-1}G_+ \cong G_+ \wedge \bigvee S^i$. Similarly, by Spanier-Whitehead duality, write $R_i = E_iG_+/E_{i-1}G_+$, note $S^0 \to \widetilde{EG} \to SEG_+$ cofiber seq. Skeleton filtration of $\widetilde{EG}$ can be

$$\begin{tikzcd} S^0 = D\widetilde{EG}^{(0)} \arrow[d] & D\widetilde{EG}^{(1)} \arrow[l] \arrow[d] & D\widetilde{EG}^{(2)} \arrow[l] \arrow[d] & \cdots \arrow[l] \\ DR_0 & DR_1 & DR_2 & \end{tikzcd}$$

$\leftarrow\!\downarrow$ cofiber seq. $D(-) =$ S-W dual

Combine to get filtration of $\widetilde{EG}$:

$$\begin{tikzcd}
\cdots \arrow[r] & F_{-2} \arrow[r] \arrow[d] & F_{-1} \arrow[r] \arrow[d] & F_0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d] & F_2 \arrow[r] \arrow[d] & \cdots \\
& Q_{-2} & Q_{-1} & Q_0 & Q_1 & Q_2 &
\end{tikzcd}$$

$\rightarrow\!\downarrow$ cofiber seq. w/ $F_k = \begin{cases} S^0 \cup CE_kG_+ = \widetilde{EG}^{(k)}, & k \geq 0 \\ D\widetilde{EG}^{(-k-1)}, & k < 0 \end{cases}$

$$Q_k = \begin{cases} SR_k, & k \geq 0 \\ DR_{-k-1}, & k < 0 \end{cases}$$

Use this to filter G-spectrum $\widetilde{EG} \wedge F(EG_+, X)$ to get the AHSS

$$E^2_{s,t}(X^{tG}) = \hat{H}^{-s}(G; \pi_t X) \overset{(*)}{\Rightarrow} \pi_{t+s}(X^{tG})$$

whose E^1-term is given by $H_{s+t}(Q_s)$.

(*) This is not convergent in general. In our case, $X = THH(\mathbb{Z}_p)$, $G = C_{p^n}$.

it is convergent. The conditional convergence is given by [Boardman].

Rk. The convergence of $E^2_{s,t}(X^{hG})$ depends on vanishing of

$$\varprojlim^{(1)} [S^n \wedge E_kG_+, X]^G = 0.$$

2.3. Relation between SSs.

Lem 1 $\exists$ surjective map from $E^r_{s,t}(X^{hG})$ to $E^r_{s,t}(X^{tG})$.

Pf. SS of X^{hG} comes from the exact couple

$$\begin{tikzcd}
\pi_*(F(E_kG_+, X), X^{hG}) \arrow[rr] & & \pi_*(F(E_{k-1}G_+, X), X^{hG}) \arrow[dl] \\
& \pi_* F(E_kG/E_{k-1}G, X) \arrow[ul] &
\end{tikzcd}$$

which converges to $\pi_* X^{hG}$. Note that (all G-equiv)

$$F(E_kG_+, X) \simeq F(E_kG_+, F(EG_+, X))$$
$$\simeq D(E_kG_+) \wedge F(EG_+, X)$$

since $D(E_kG_+) = F(E_kG_+, S^0)$. By cofiber seqs

$$S^0 \longrightarrow \widetilde{EG}^{(k)} \longrightarrow S^1 \wedge E_{k-1}G_+$$

$$F_{-k-1} = D\widetilde{EG}^{(k)} \longrightarrow S^0 \longrightarrow D(E_{k-1}G_+)$$

Thus $\Sigma^{-1}\left(F(E_{k-1}G_+, X)/F(EG_+, X)\right) \simeq \Sigma^{-1}(D(E_{k-1}G_+))/S^0 \wedge F(EG_+, X)$

$$\simeq D\widetilde{EG}^{(k)} \wedge F(EG_+, X)$$
$$\simeq F_{-k-1} \wedge F(EG_+, X).$$

$\Rightarrow$ corresponding to filtration $\widetilde{EG} \wedge F(EG_+, X)$, leading to $E^2_{s,t}(X^{tG})$.

- From $EG_+ \wedge X \simeq EG_+ \wedge F(EG_+, X)$

$$E_kG_+ / E_{k-1}G_+ = \Sigma^{-1} F_{k+1}/F_k .$$

$$\partial : F_{s+1} \wedge F(EG_+, X) \longrightarrow \begin{cases} F_{s+1}/F_0 \wedge F(EG_+, X), & s \geq 0 \\ *, & s < 0 \end{cases}$$

induces a map of SSs:

$$\partial_* : E^r_{s+1,t}(X^{tG}) \longrightarrow E^r_{s,t}(X_{hG})$$

s.t. ∂_* injective, $s \geq 0$, $r \geq 2$. On E^∞-page it's associated to the natural map from $\Sigma^{-1} X^{tG}$ to X_{hG}.

Now, $\forall \alpha \in E^\infty_{s,t}(X^{hG})$, $s \geq 0$, $\alpha \in \ker\left(E^\infty_{-s,t}(\psi) : E^\infty_{-s,t}(X^{hG}) \to E^\infty_{-s,t}(X^{tG})\right)$ if $\exists\ r > s$ and $\beta \in E^r_{r-s,t-r+1}(X^{tG})$ w/ $d^r(\beta) = \alpha$. Now

$$\partial_* : E^r_{r-s,t-r+1}(X^{tG}) \longrightarrow E^r_{r-s-1,t-r+1}(X_{hG})$$

injective, β mapped into $E^\infty(X_{hG})$. Then one can show β survives, and so it represents an elt of $\pi_{t-s}(X_{hG})$, which is mapped to a rep. of α by N.

Lem 2 $\alpha \in \ker \cdots \cdots$ $s \geq 0$. Then $\exists\, r > s$ s.t. $\alpha \in \operatorname{im} d^r$, where

$$d^r : E^r_{r-s,\, t-r+1}(X^{tG}) \longrightarrow E^r_{-s,t}(X^{tG}).$$

Moreover, if $d^r(\beta) = \alpha$, then $\partial_* \beta$ survives to $E^\infty_{r-s-1,\, t-r+1}(X_{hG})$, and is rep. by an elet of $\pi_{t-s}(X_{hG})$, s.t. with good choice, its image under norm map is rep. by $\alpha \in E^\infty_{-s,t}(X^{hG})$.

Rk. $E^2_{s,t}(X^{tG}) = E^2_{s,t}(X^{hG})$, $s < 0$

$E^2_{s+1,t}(X^{tG}) = E^2_{s,t}(X_{hG})$, $s \geq 1$.

For $s = 0, 1$, s.e.s.

$$0 \to \hat{H}^{-1}(G; \pi_t X) \longrightarrow H_0(G; \pi_t X) \xrightarrow{\text{norm}} H^0(G; \pi_t X) \to \hat{H}^0(G; \pi_t X) \to 0.$$

3. Witt rings.

Let A be comm. ring. $W(A) := A^{\aleph_0}$ as sets. $a \in W(A)$. It has the form

$a = (a_0, a_1, \cdots)$

Def A ghost map is $w : W(A) \longrightarrow A^{\aleph_0}$ given by

$$a \longmapsto (w_0(a), w_1(a), \cdots)$$

w/ w_i Witt poly. (or Witt vectors)

$w_0 = a_0$

$w_1 = a_0^p + p a_1$

$w_2 = a_0^{p^2} + p a_1^p + p^2 a_2$

$\vdots$

$w_n = a_0^{p^n} + p a_1^{p^{n-1}} + \cdots + p^n a_n$

- If we define $a + b = (s_0(a,b), s_1(a,b), \cdots)$

  $ab = (p_0(a,b), p_1(a,b), \cdots)$

for some s_i, p_i depends on $a_0, \ldots, a_i, b_0, \ldots, b_i$ polys. Then $W(A)$ is a ring whence some terms $S_0(a,b) = a_0 + b_0$

$S_1(a,b) = (a_0^p + b_0^p - (a_0+b_0)^p)/p$

$P_0(a,b) = a_0 b_0$

$P_1(a,b) = a_1 b_0^p + p a_1 b_1 + a_0^p b_1$

Hence $(W(A), +, \cdot)$ is a comm. ring w/ $0_{W(A)} = (0, \dots, 0, \dots)$

$1_{W(A)} = (1, 0, 0, \dots, 0, \dots)$

- Frobenius homomorphism $F: W(A) \longrightarrow W(A)$

  $(w_0, w_1, \dots) \longmapsto (w_1, w_2, \dots)$

  this is a ring homomphism.

  Verschiebung map $V: W(A) \longrightarrow W(A)$

  $(a_0, a_1, \dots) \longmapsto (0, a_0, a_1, \dots)$

  this is an additive map.

  Teichmüller character $r: A \longrightarrow W(A)$

  $a \longmapsto (a, 0, 0, \dots)$

  this is a multiplicative map.

<u>Cor</u>. A $\mathbb{F}_p$-alg. $F = W(\phi)$. $\phi: a \mapsto a^p$. $a \in A$.

Note $\forall a \in W(A)$, $a = (a_0, a_1, \dots) = \sum_n V^n(r(a_n))$. $V^n = \underbrace{V \circ \cdots \circ V}_{n}$

<u>Cor</u>. $aV(b) = V(F(a)b)$. $FV(a) = pa$, $VF(a) = V(1_{W(A)})a$

$V^n W(A) =$ ideal of $W(A)$

<u>Notation</u> $W_n A = W(A)/V^n W(A)$. ring of Witt vectors of length n.

Write $R: W_{n+1}(A) \longrightarrow W_n(A)$ be the restriction map. We have a s.e.s.

$(a_0, \dots, a_{n+1}) \longmapsto (a_0, \dots, a_n)$

$$0 \longrightarrow W_n A \xrightarrow{V^r} W_{n+r} A \xrightarrow{R^n} W_r A \longrightarrow 0.$$

Thm k perfect field. char $k = p$, then $W(k)$ = discrete valuation w/ max ideal gen. by p. In particular $W(\mathbb{F}_p) = \mathbb{Z}_p$. p-adic numbers

4. Properties of cyclotomic trace.

Write $K_i(A;\mathbb{Z}_p) = \pi_i(K(A)^\wedge_p)$. $TC_i(A;\mathbb{Z}_p) = \pi_i(TC(A)^\wedge_p)$.

Theorem 1 For finite $W(k)$-alg, k perfect field, char $k = p$, the cycl. trace

$$Trc : K_i(A;\mathbb{Z}_p) \longrightarrow TC_i(A;\mathbb{Z}_p)$$

is iso, $i \geq 0$.

Note A finite over $W(k)$. $A = \operatorname{colim} A/p^nA$. $W(k)$ PID, p-adically complete.

Define $K^{top}(A) = \underleftarrow{\text{holim}}\, K(A/p^nA)$

$TC^{top}(A) = \underleftarrow{\text{holim}}\, TC(A/p^nA)$

Then one has s.e.s.

$$0 \to \underleftarrow{\lim}^{(1)} K_{i+1}(A/p^nA;\mathbb{Z}_p) \to K_i^{top}(A;\mathbb{Z}_p) \to \underleftarrow{\lim}\, K_i(A/p^nA;\mathbb{Z}_p) \to 0$$

$$0 \to \underleftarrow{\lim}^{(1)} TC_{i+1}(A/p^nA;\mathbb{Z}_p) \to TC_i^{top}(A;\mathbb{Z}_p) \to \underleftarrow{\lim}\, TC_i(A/p^nA;\mathbb{Z}_p) \to 0$$

Theorem 2 (McCarthy) Let $R \to S$ be a surjection of rings w/ nilpotent kernel.

Then the diagrams

$$\begin{tikzcd} K(R)^\wedge \arrow[r] \arrow[d] & TC(R)^\wedge \arrow[d] \\ K(S)^\wedge \arrow[r] & TC(S)^\wedge \end{tikzcd}$$

of profinitely completed spectra is htpy Cartesian (pullback). In particular

$$K(R \to S)^\wedge \sim TC(R \to S)^\wedge$$

Theorem 3 (Dundas)

$f : L_1 \to L_2$ map of FSPs w/ $\pi_0 f$ surjective, $\ker \pi_0 f$ nilpotent

Then the diagram

$$\begin{tikzcd} K(L_1)^\wedge \arrow[r] \arrow[d] & TC(L_1)^\wedge \arrow[d] \\ K(L_2)^\wedge \arrow[r] & TC(L_2)^\wedge \end{tikzcd}$$

is htpy Cartesian.

Rk Proof of Theorem 2 & 3 uses the trick of Goodwillie's calculus of functors.

IDEA of pf of Theorem 1 Equivalent to prove

① $K_i(A/pA ; \mathbb{Z}_p) \xrightarrow{\cong} TC_i(A/pA ; \mathbb{Z}_p)$. $i \geq 0$

② $TC_i(A; \mathbb{Z}_p) \xrightarrow{\cong} TC_i^{top}(A ; \mathbb{Z}_p)$. $i \geq 0$

③ $K_i(A; \mathbb{Z}_p) \xrightarrow{\cong} K_i^{top}(A ; \mathbb{Z}_p)$. $i \geq 0$.

Given ① + Theorem 2 , $K_i(A/p^nA ; \mathbb{Z}_p) \xrightarrow{\cong} TC_i(A/p^nA ; \mathbb{Z}_p)$. $i \geq 0$.

By s.e.s. $\Rightarrow$ $K_i^{top}(A; \mathbb{Z}_p) \xrightarrow{\cong} TC_i^{top}(A; \mathbb{Z}_p)$. $i \geq 0$

Use ② . ③ $\Rightarrow$ ✓.

Theorem 4 Let L be a FSP s.t. $\pi_0 L = \varinjlim \pi_n L(S^n)$ is a finite $W(k)$-alg. for some perfect field k w/ char $k = p$. Then

$$Trc : K(L)^\wedge_p \longrightarrow TC(L)^\wedge_p$$

is a htpy equiv.

pf. Theorem 3 + Theorem 1.

5. Cyclotomic Spectra and Basic Results.

Let $\rho_p : S^1 \to S^1/C_p$. If $X \in S^1\text{-}Top$. then $X^{C_p} \in S^1/C_p - Top$

$z \mapsto \sqrt[p]{z}$

ρ_p iso.

Use ρ_p, can view S^1/C_p - spectra as S^1 - spectra. Explicitly, given E indexed on $\mathcal{U}^{C_p}$ ($\mathcal{U}$ complete universe of S^1), we have S^1 - spectrum $\rho_p^{\#}E$ indexed on $\rho_p^*\mathcal{U}^{C_p}$ w/

- $\rho_p^*\mathcal{U}^{C_p} = \mathcal{U}$, ρ_p^* change-of-universe
- $\rho_p^{\#}E(V) = \rho_p^* E((\rho_p^{-1})^*(V))$. $V \subset \mathcal{U}$.

Def. A cyclotomic spectrum is an S^1 - spectrum indexed on $\mathcal{U}$ w/ an S^1 - equiv

$$\eta_C : \rho_C^{\#}\Phi^C X \to X$$

for every finite subgp $C \subset S^1$ s.t. $\forall$ pair of finite subgps the diagram commutes:

$$\begin{tikzcd}
\rho_{C_r}^{\#}\Phi^{C_r}\rho_{C_s}^{\#}\Phi^{C_s}X \arrow[r, "="] \arrow[d, "\rho_{C_r}^{\#}\Phi^{C_r}\eta_{C_s}"'] & \rho_{C_{rs}}^{\#}\Phi^{C_{rs}}X \arrow[d, "\eta_{C_s}"] \\
\rho_{C_r}^{\#}\Phi^{C_r}X \arrow[r, "\eta_{C_r}"] & X
\end{tikzcd}$$

If $C = C_p$, then call it p-cyclotomic spectrum (p prime).

Thm $\forall$ FSP F, tHH(F) is a p-cyclotomic spectrum. Abuse the notation, we also denote it by THH(F). Here

$$THH(F)(V) = THH(F; S^V) = \left([p] \mapsto \underrightarrow{\mathrm{holim}}_{(\Delta)^p} Map(S^{i_0}\wedge \cdots \wedge S^{i_p}, F(S^{i_0})\wedge \cdots \wedge F(S^{i_p}) \wedge S^V) \right).$$

Equivalently, tHH is a (p-) cyclotomic spectrum whose underlying naive S^1 - spectrum is THH.

Thm If X is a p-cyclotomic spectrum, then $\Phi^{C_{p^n}} X \simeq X^{C_{p^{n-1}}}$.

▲ This follows from

Lem $X, Y \in C_{p^n}\text{-}\mathrm{Top}_*$. Then $\exists$ C_{p^n}/C_p - htpy equiv.

$$F(X, Y \wedge \widetilde{ES^1})^{C_p} \simeq F(X^{C_p}, Y^{C_p}).$$

Now, $X = THH(F)$, $G = C_{p^n}$, we get

$$\begin{tikzcd}
THH(F)_{hC_{p^n}} \arrow[r] \arrow[d, equal] & THH(F)^{C_{p^n}} \arrow[r] \arrow[d, "\gamma_n"] & THH(F)^{C_{p^{n-1}}} \arrow[d, "\hat{\gamma}_n"] \\
THH(F)_{hC_{p^n}} \arrow[r, "N"] & THH(F)^{hC_{p^n}} \arrow[r] & THH(F)^{tC_{p^n}}
\end{tikzcd}$$

The following are known results : (Mainly by Bökstedt, Madsen)

Let $F_R =$ FSP associated to ring R, i.e.

$$F_R(S) = |R\Delta_\cdot(S) / R\Delta_\cdot(*)|$$

Then $F_R(S^n) = K(R, n)$, K-theory of F_R is htpy equiv to $BGL(R)^+ \times \mathbb{Z}$.

Thus, can regard F_R as R. Write $THH(R)$ for $THH(F_R)$.

$TC(R, p)$ for $TC(F_R, p)$.

Theorem 1 (Bökstede) $\pi_*(THH(\mathbb{Z}_p); \mathbb{F}_p) \cong E\{e\} \otimes \mathbb{F}_p[f]$, where

$\deg e = 2p-1$, $\deg f = 2p$, $\beta(e) = f$, where β = Bockstein homophism.

E = exterior alg.

Theorem 2 (Bökstede - Madsen, § 4) $\pi_*(THH(\mathbb{Z}_p)^{tC_p}; \mathbb{F}_p) = E\{e\} \otimes \mathbb{F}_p[t^p, t^{-p}]$

- (B-M, Lem 6.5) $i \geq 0$, $\hat{\gamma}_{i*} : \pi_i(THH(\mathbb{Z}_p); \mathbb{F}_p) \xrightarrow{\cong} \pi_i(THH(\mathbb{Z}_p)^{tC_p}; \mathbb{F}_p)$

What's more, (B-M, Lem 6.7), $i \geq 0$,

$$\hat{\gamma}_* : \pi_i(THH(\mathbb{Z}_p)^{C_{p^{n-1}}}; \mathbb{F}_p) \xrightarrow{\cong} \pi_i(THH(\mathbb{Z}_p)^{tC_{p^{n-1}}}; \mathbb{F}_p)$$

$$\gamma_* : \pi_i(THH(\mathbb{Z}_p)^{C_{p^n}}; \mathbb{F}_p) \xrightarrow{\cong} \pi_i(THH(\mathbb{Z}_p)^{hC_{p^n}}; \mathbb{F}_p)$$

Theorem 3 (B-M. Thm 7.15)

$$\pi_{2r-1}(TC(\mathbb{Z}_p, p); \mathbb{F}_p) = \begin{cases} \mathbb{F}_p, & r \not\equiv 0, 1 \bmod (p-1) \text{ or } r=1 \\ \mathbb{F}_p \oplus \mathbb{F}_p, & \text{else} \end{cases}$$

$$\pi_{2r}(TC(\mathbb{Z}_p, p); \mathbb{F}_p) = \begin{cases} \mathbb{F}_p \oplus \mathbb{F}_p, & r \equiv 0 \bmod (p-1), \ r \neq 0 \\ \mathbb{F}_p, & r=0 \\ 0, & \text{else.} \end{cases}$$

- Application in algebraic K-thy:

Theorem 4 (Hesselholt - Madsen)

k perfect field, char $k = p > 0$. Then

$$K_{2m-1}(k[x]/x^n ; \mathbb{Z}_p) = W_{nm-1}(k) / V^n W_{m-1}(k),$$

$$K_{2m}(\cdot\cdot \quad \cdot\cdot) = 0, \ \forall\, m > 0.$$

W = Witt rings, $W_n = W/V^n W$. V = Verschiebung map.

Pf Sketch. By Property of cyclotomic trace, $K_i(\cdot\cdot ; \mathbb{Z}_p) \cong TC_i(\cdot\cdot ; \mathbb{Z}_p)$. $i \geq 0$. To calculate the latter. First to use the fact that, the htpy fiber of $TC(k[x]/(x^n)) \longrightarrow TC(k)$, denoted $\widetilde{TC}(k[x]/(x^n))$, which is the "reduced" version of $TC(k[x]/(x^n))$, is equivalent to product of some very complicated spectra, which passing to htpy gps will be iso to $W_{m'}(k)$ for some good m'. Deduce the result from this fact.

Ref [Madsen, Algebraic K-theory and traces, Theorem 5.2.6. 5.2.7, 5.2.8]

IV. Remarks on new approach (N-S. 2018)

N-S gave a new definition on cyclotomic spectra. Namely, $X \in CycSp$ is an object in ∞-cat of Sp, which is S^1-equivariant, w/ $S^1/C_p \cong S^1$-equivariant map $\varphi_p : X \rightarrow X^{tC_p}$, $\forall$ p prime, where $X^{tC_p} = \mathrm{cofib}(X_{hC_p} \rightarrow X^{hC_p})$

The main theorem is $CycSp^{gen} \cong CycSp$ (on bounded below spectra), and so TC becomes an equalizer of some pair of maps.

This tells us, the only extra info stored in geninne cycl. spectra is this Frobenius map. Note that $CycSp^{gen}$, geninne cycl. spectra, is the traditional definition of cycl. spectra. The equivalence is up to ∞- htpy coherent.